Turbulent motions in the Atmosphere and Oceans

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Course description

The course will present the phenomena, theory, and modeling of turbulence in the Earth's oceans and atmosphere. The scope will range from centimeter to planetary scale motions. The regimes of turbulence will include homogeneous isotropic three dimensional turbulence, convection, boundary layer turbulence, internal waves, two dimensional turbulence, quasi-geostrophic turbulence, and planetary scale motions in the ocean and atmosphere. Prerequisites: the mathematics and physics required for admission to the graduate curriculum in the EAPS department, or consent of the instructor.

Course requirements

Class attendance and discussion, weekly homework assignments.

Reference texts

Andrews, Holton, and Leovy, "Middle atmosphere dynamics"
Frisch, "Turbulence: the legacy of Kolmogorov"
Lesieur, "Turbulence in Fluids", 3rd revised edition
McComb, "The physics of turbulence"
Saffman, "Vortex dynamics"
Salmon, "Lectures on geophysical fluid dynamics"
Tennekes and Lumley, "A first course in Turbulence"
Whitham, "Linear and nonlinear waves"

Chapter 3

Isotropic homogeneous 3D turbulence

Turbulence was recognized as a distinct fluid behavior by Leonardo da Vinci more than 500 years ago. It is Leonardo who termed such motions "turbolenze", and hence the origin of our modern word for this type of fluid flow. But it wasn't until the beginning of last century that researchers were able to develop a rigorous mathematical treatment of turbulence. The first major step was taken by G. I. Taylor during the 1930s. Taylor introduced formal statistical methods involving correlations, Fourier transforms and power spectra into the turbulence literature. In a paper published in 1935 in the Proceedings of the Royal Society of London, he very explicitly presents the assumption that turbulence is a random phenomenon and then proceeds to introduce statistical tools for the analysis of homogeneous, isotropic turbulence. In 1941 the Russian statistician A. N. Kolmogorov published three papers (in Russian) that provide some of the most important and most-often quoted results of turbulence theory. These results, which will be discussed in some detail later, comprise what is now referred to as the K41 theory, and represent a major success of the statistical theories of turbulence. This theory provides a prediction for the energy spectrum of a 3D isotropic homogeneous turbulent flow. Kolmogorov proved that even though the velocity of an isotropic homogeneous turbulent flow fluctuates in an unpredictable fashion, the energy spectrum (how much kinetic energy is present on average at a particular scale) is predictable.

The spectral theory of Kolmogorov had a profound impact on the field and it still represents the foundation of many theories of turbulence. It it thus appropriate to start this course by introducing the concepts of 3D isotropic homogeneous turbulence and K41. It should however be kept in mind that 3D isotropic homogeneous turbulence is an idealization never encountered in nature. The challenge is then to understand what aspects of these theories apply to natural flows and what are pathological.

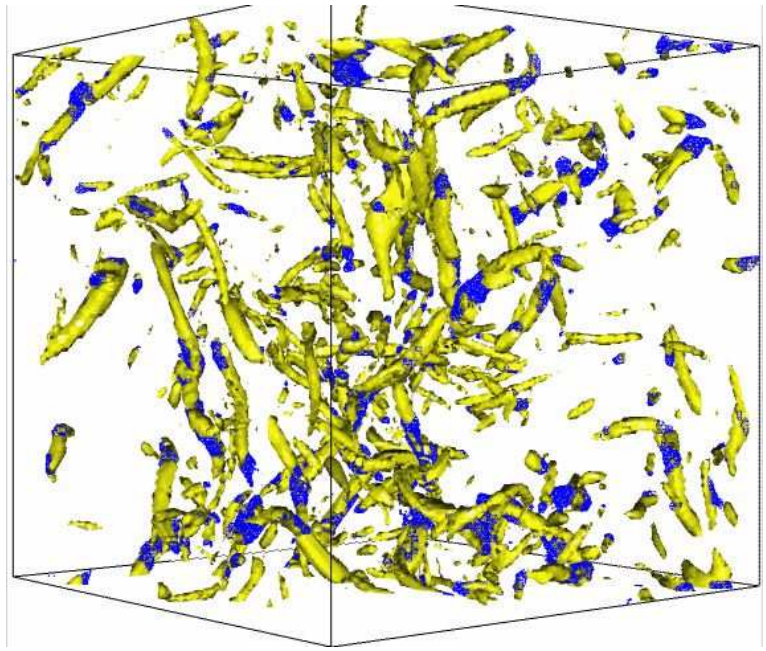

Figure 3.1: Isosurfaces of the the velocity gradient tensor used to visualize structures in computation of isotropic homogeneous 3D turbulence. The yellow surfaces represent flow regions with stable focus/stretching topology while the blue outlines of the isosurfaces show regions with unstable focus/contracting topology. 128^3 simulation with Taylor Reynolds number = 70.9. (Andrew Ooi, University of Melbourne, Australia, 2004, http://www.mame.mu.oz.au/fluids/).

A turbulent flow is said to be *isotropic* if,

- rotation and buoyancy are not important and can be neglected,
- there is no mean flow.

Rotation and buoyancy forces tend to suppress vertical motions, as we discuss later in the course, and create an anisotropy between the vertical and the horizontal directions. The presence of a mean flow with a particular orientation can also introduce anisotropies in the turbulent velocity and pressure fields.

A flow is said to be *homogeneous* if,

- there are no spatial gradients in any averaged quantity.

This is equivalent to assume that the statistics of the turbulent flow is not a function of space. An example of 3D isotropic homogeneous flow is shown in Fig. 3.1.

The theory of 3D isotropic homogeneous turbulence is based on the examination of the kinetic energy budget (potential energy is constant for flows with no buoyancy fluctuations),

$$\frac{\partial}{\partial t}\left(\frac{u_i^2}{2}\right)+\frac{\partial}{\partial x_j}\left(u_j\frac{u_i^2}{2}-\nu u_i\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial}{\partial x_i}\left(\frac{pu_i}{\rho_0}\right)-\nu\left(\frac{\partial u_i}{\partial x_j}\right)^2. \tag{3.1}$$

If we take an ensemble average of this equation under the assumptions of homogeneity we get,

$$\frac{\mathrm{dE}}{\mathrm{dt}}=-\epsilon, \tag{3.2}$$

where,

$$E\equiv\langle\frac{u_i^2}{2}\rangle,\qquad \epsilon\equiv\langle\nu\left(\frac{\partial u_i}{\partial x_j}\right)^2\rangle. \tag{3.3}$$

This equation state that the rate of change of turbulent kinetic energy E (TKE) is balanced by viscous dissipation ϵ. Such a balance cannot be sustained for long times - a source of kinetic energy is needed. However sources of TKE are typically not homogeneous: think of a stirrer or an oscillating boundary. We sidestep this contradiction by assuming that for large Reynolds numbers, although isotropy and homogeneity are violated by the mechanism producing the turbulence, they still hold at small scales and away from boundaries. Then the turbulence production can be represented simply by a forcing term F, assumed to be isotropic and homogeneous:

$$\frac{\mathrm{dE}}{dt}=-\epsilon+F. \tag{3.4}$$

3.1 Kinetic Energy Spectra for 3D turbulence

Definition of KE in spectral space

For a flow which is homogeneous in space (i.e. statistical properties are independent of position), a spectral description is very appropriate, allowing us to examine properties as a function of wavelength. The total kinetic energy can be written replacing the ensemble average with a space average,

$$E=\frac{1}{2}\langle u_i^2\rangle=\frac{1}{2}\frac{1}{V}\iiint u_i(\mathbf{x})u_i(\mathbf{x})\ \mathrm{d}\mathbf{x}, \tag{3.5}$$

where V is the volume domain. The spectrum $\phi_{i,j}(\mathbf{k})$ is then defined by,

$$E=\frac{1}{2}\iiint\phi_{i,i}(\mathbf{k})\mathrm{d}\mathbf{k}=\iiint E(\mathbf{k})\mathrm{d}\mathbf{k} \tag{3.6}$$

where $\phi_{i,j}(\mathbf{k})$ is the Fourier transform of the velocity correlation tensor $R_{i,j}(\mathbf{r})$,

$$\phi_{i,j}(\mathbf{k}) = \frac{1}{(2\pi)^3}\iiint R_{i,j}(\mathbf{r})e^{-\mathrm{i}\mathbf{k}.\mathbf{r}}\mathrm{d}\mathbf{r}, \qquad R_{i,j}(\mathbf{r}) = \frac{1}{V}\iiint u_j(\mathbf{x})u_i(\mathbf{x}+\mathbf{r})\mathrm{d}\mathbf{x}. \quad (3.7)$$

$R_{i,j}(\mathbf{r})$ tells us how velocities at points separated by a vector $\mathbf{r}$ are related. If we know these two point velocity correlations, we can deduce $E(\mathbf{k})$. Hence the energy spectrum has the information content of the two-point correlation.

$E(\mathbf{k})$ contains directional information. More usually, we want to know the energy at a particular scale $k = |\mathbf{k}|$ without any interest in separating it by direction. To find $E(k)$, we integrate over the spherical shell of radius k (in 3-dimensions),

$$E = \iiint E(\mathbf{k})\mathrm{d}\mathbf{k} = \int_0^\infty \left[\oint k^2 E(\mathbf{k})d\sigma\right]\mathrm{d}k = \int_0^\infty E(k)\mathrm{d}k, \quad (3.8)$$

where σ is the solid angle in wavenumber space, i.e. $\mathrm{d}\sigma = \sin\theta_1\ \mathrm{d}\theta_1\ \mathrm{d}\theta_2$. We now define the isotropic spectrum as,

$$E(k) = \oint k^2 E(\mathbf{k})\mathrm{d}\sigma = \frac{1}{2}\oint k^2\phi_{i,i}(\mathbf{k})\mathrm{d}\sigma. \quad (3.9)$$

For isotropic velocity fields the spectrum does not depend on directions, i.e. $\phi_{i,i}(\mathbf{k}) = \phi_{i,i}(k)$, and we have,

$$E(k) = 2\pi k^2\phi_{i,i}(k). \quad (3.10)$$

Energy budget equation in spectral space

We have an equation for the evolution of the total kinetic energy E. Equally interesting is the evolution of $E(k)$, the isotropic energy at a particular wavenumber k. This will include terms which describe the transfer of energy from one scale to another, via nonlinear interactions.

To obtain such an equation we must take the Fourier transform of the non-rotating, unstratified Boussinesq equations,

$$\frac{\partial u_i}{\partial t} - \nu\frac{\partial^2 u_i}{\partial x_j^2} = -u_j\frac{\partial u_i}{\partial x_j} - \frac{1}{\rho_0}\frac{\partial p}{\partial x_i}. \quad (3.11)$$

The two terms on the lhs are linear and are easily transformed into Fourier space,

$$\frac{\partial}{\partial t}u_i(\mathbf{x},t) \iff \frac{\partial}{\partial t}\hat{u}_i(\mathbf{k},t), \quad (3.12)$$

$$\nu\frac{\partial^2}{\partial x_j^2}u_i(\mathbf{x},t) \iff -\nu k_j^2\hat{u}_i(\mathbf{k},t). \quad (3.13)$$

In order to convert the pressure gradient term, we first notice that taking the divergence of the Navier-Stokes equation we obtain,

$$\frac{\partial^2 p}{\partial x_i^2} = -\rho_0 \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}. \tag{3.14}$$

Thus both terms on the rhs of eq. (3.11) involve the product of velocities. The convolution theorem states that the Fourier transform of a product of two functions is given by the convolution of their Fourier transforms,

$$\frac{1}{V}\iiint u_i(\mathbf{x},t)u_j(\mathbf{x},t)e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{x} = \frac{1}{(2\pi)^3}\iiint \hat{u}_i(\mathbf{p},t)\hat{u}_j(\mathbf{q},t)\delta(\mathbf{p}+\mathbf{q}-\mathbf{k})\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}. \tag{3.15}$$

Applying the convolution terms to the terms on the rhs we get, The two terms on the lhs are linear and are easily transformed in Fourier space,

$$u_j\frac{\partial u_i}{\partial x_j} \iff \mathrm{i}\iiint q_j\hat{u}_j(\mathbf{p},t)\hat{u}_i(\mathbf{q},t)\delta(\mathbf{p}+\mathbf{q}-\mathbf{k})\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}, \tag{3.16}$$

$$p \iff \rho_0\iiint \frac{p_i q_j}{k^2}\hat{u}_j(\mathbf{p},t)\hat{u}_i(\mathbf{q},t)\delta(\mathbf{p}+\mathbf{q}-\mathbf{k})\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}. \tag{3.17}$$

Plugging all these expressions in eq. (3.11) we obtain the Navier-Stokes equation in Fourier space,

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\hat{u}_i(\mathbf{k},t) = -\mathrm{i}\iiint q_j\left(\delta_{i,m}-\frac{k_i p_m}{k^2}\right)\hat{u}_j(\mathbf{p},t)\hat{u}_m(\mathbf{q},t)\delta(\mathbf{p}+\mathbf{q}-\mathbf{k})\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q} \tag{3.18}$$

The term on the right hand side shows that the nonlinear terms involve triad interactions between wave vectors such that $\mathbf{k}=\mathbf{p}+\mathbf{q}$.

Now to obtain the energy equation we multiply e. (3.18) by $\hat{u}_i^*(\mathbf{k},t)$ and we integrate over $\mathbf{k}$,

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)\phi_{i,i}(\mathbf{k},t) = \mathrm{Re}\left[\iiint A_{ijm}(\mathbf{k},\mathbf{p},\mathbf{q})\hat{u}_i^*(\mathbf{k},t)\hat{u}_j(\mathbf{p},t)\hat{u}_m(\mathbf{q},t)\delta(\mathbf{p}+\mathbf{q}-\mathbf{k})\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}\mathrm{d}\mathbf{k}\right]. \tag{3.19}$$

The terms on the rhs represent the triad interactions that exchange energy between $\hat{u}_i(\mathbf{k},t)$, $\hat{u}_j(\mathbf{p},t)$, and $\hat{u}_m(\mathbf{q},t)$. The coefficient A_{ijm} are the coupling coefficient of each triad and depends only on the wavenumbers.

If pressure and advection were not present, the energy equation would reduce to,

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)\phi_{i,i}(\mathbf{k},t) = 0, \tag{3.20}$$

in which the wavenumbers are uncoupled. The solution to this equation is,

$$\phi_{i,i}(\mathbf{k},t) = \phi_{i,i}(\mathbf{k},0)e^{-\nu k^2 t}. \tag{3.21}$$

According to (3.21), the energy in wavenumber $\mathbf{k}$ decays exponentially, at a rate that increases with increasing wavenumber magnitude k. Thus viscosity damps the smallest spatial scale fastest.

We now restrict our analysis to isotropic velocity fields, so that we can use (3.10) and simplify (3.19),

$$\frac{\partial}{\partial t}E(k,t) = T(k,t) - 2\nu k^2 E(k,t), \tag{3.22}$$

where $T(k,t)$ comprises all triad interaction terms. If we examine the integral of this equation over all k,

$$\frac{\partial}{\partial t}\int_0^\infty E(k)\mathrm{d}k = \int_0^\infty T(k,t)\mathrm{d}k - 2\nu\int_0^\infty k^2 E(k)\mathrm{d}k, \tag{3.23}$$

and note that $-2\nu k^2 E(k)$ is the Fourier transform of the dissipation term, then we see that the equation for the total energy budget in (3.2), is recovered only if,

$$\int_0^\infty T(k,t)\mathrm{d}k = 0. \tag{3.24}$$

Hence the nonlinear interactions transfer energy between different wave numbers, but do not change the total energy.

Now, adding a forcing term to the energy equation in k-space we have the following equation for energy at a particular wavenumber k,

$$\frac{\partial}{\partial t}E(k,t) = T(k,t) + F(k,t) - 2\nu k^2 E(k,t), \tag{3.25}$$

where $F(k,t)$ is the forcing term, and $T(k,t)$ is the **kinetic energy transfer**, due to nonlinear interactions. The **kinetic energy flux** through wave number k is $\Pi(k,t)$, defined as,

$$\Pi(k,t) = \int_k^\infty T(k',t)\mathrm{d}k', \tag{3.26}$$

or

$$T(k,t) = -\frac{\partial \Pi(k,t)}{\partial k}. \tag{3.27}$$

For stationary turbulence,

$$2\nu k^2 E(k) = T(k) + F(k). \tag{3.28}$$

Remembering that the total dissipation rate is given by,

$$\epsilon = \int_0^\infty 2\nu k^2 E(k)\mathrm{d}k \tag{3.29}$$

and that the integral of the triad interactions over the whole k-space vanishes, we have,

$$\epsilon = \int_0^\infty F(k)\mathrm{d}k. \tag{3.30}$$

The rate of dissipation of energy is equal to the rate of injection of energy.

If the forcing $F(k)$ is concentrated on a narrow spectral band centered around a wave number k_i, then for $k \neq k_i$,

$$2\nu k^2 E(k) = T(k). \tag{3.31}$$

In the limit of $\nu \to 0$, the energy dissipation becomes negligible at large scales. Thus there must be an intermediate range of scales between the forcing scale and the scale where viscous dissipation becomes important, where,

$$2\nu k^2 E(k) = T(k) \approx 0. \tag{3.32}$$

Notice that ϵ must remain nonzero, for nonzero $F(k)$, in order to balance the energy injection. This is achieved by $\int_0^\infty k^2 E(k)\mathrm{d}k \to \infty$, i.e. the velocity fluctuations at small scales increase.

Then we find the energy flux in the limit $\nu \to 0$,

$$\begin{aligned} \Pi(k) &= 0, \ : \ k < k_i \\ \Pi(k) &= \epsilon \ : \ k > k_i \end{aligned} \tag{3.33}$$

Hence at vanishing viscosity, the kinetic energy flux is constant and equal to the injection rate, for wavenumbers greater than the injection wavenumber k_i. The scenario is as follows. (a) Energy is input at a rate ϵ at a wavenumber k_i. (b) Energy is fluxed to higher wavenumbers at a rate ϵ trough triad interactions. (c) Energy is eventually dissipated at very high wavenumbers at a rate ϵ, even in the limit of $\nu \to 0$.

The statement that triad interactions produce a finite energy flux ϵ toward small scales does not mean that all triad interactions transfer energy exclusively toward small scales. Triad interactions transfer large amounts of energy toward both large and small scales. On average, however, there is an excess of energy transfer toward small scales given by ϵ.

Kolmogorov spectrum

Kolmogorov's 1941 theory for the energy spectrum makes use of the result that ϵ, the energy injection rate, and dissipation rate also controls the flux of energy. Energy flux is independent of wavenumber k, and equal to ϵ for $k > k_i$. Kolmogorov's theory assumes the injection wavenumber is much less than the dissipation wavenumber ($k_i << k_d$, or large Re). In the intermediate range of scales $k_i < k < k_d$ neither the

forcing nor the viscosity are explicitly important, but instead the energy flux ϵ and the local wavenumber k are the only controlling parameters. Then we can express the energy density as

$$E(k) = f(\epsilon, k) \tag{3.34}$$

Now using dimensional analysis:

Quantity	Dimension
Wavenumber k	$1/L$
Energy per unit mass E	$U^2 \sim L^2/T^2$
Energy spectrum $E(k)$	$EL \sim L^3/T^2$
Energy flux ϵ	$E/T \sim L^2/T^3$

In eq. (3.34) the lhs has dimensionality L^3/T^2; the dimension T^{-2} can only be balanced by $\epsilon^{2/3}$ because k has no time dependence. Thus,

$$E(k) = \epsilon^{2/3} g(k). \tag{3.35}$$

Now $g(k)$ must have dimensions $L^{5/3}$ and the functional dependence we must have, if the assumptions hold, is,

$$E(k) = C_K \epsilon^{2/3} k^{-5/3} \tag{3.36}$$

This is the famous *Kolmogorov spectrum*, one of the cornerstone of turbulence theory. C_K is a universal constant, the *Kolmogorov constant*, experimentally found to be approximately 1.5. The region of parameter space in k where the energy spectrum follows this $k^{-5/3}$ form is known as the **inertial range**. In this range, energy **cascades** from the larger scales where it was injected ultimately to the dissipation scale. The theory assumes that the spectrum at any particular k depends only on spectrally local quantities - i.e. has no dependence on k_i for example. Hence the possibility for long-range interactions is ignored.

We can also derive the Kolmogorov spectrum in a perhaps more physical way (after Obukhov). Define an eddy turnover time $\tau(k)$ at wavenumber k as the time taken for a parcel with energy $E(k)$ to move a distance $1/k$. If $\tau(k)$ depends only on $E(k)$ and k then, from dimensional analysis,

$$\tau(k) \sim \left[k^3 E(k)\right]^{-1/2} \tag{3.37}$$

The energy flux can be defined as the available energy divided by the characteristic time τ. The available energy at a wavenumber k is of the order of $kE(k)$. Then we have,

$$\epsilon \sim \frac{kE(k)}{\tau(k)} \sim k^{5/2} E(k)^{3/2}, \tag{3.38}$$

and hence,

$$E(k) \sim \epsilon^{2/3} k^{-5/3}. \tag{3.39}$$

Characteristic scales of turbulence

Kolmogorov scale

We have shown that viscous dissipation acts most efficiently at small scales. Thus above a certain wavenumber k_d, viscosity will become important, and $E(k)$ will decay more rapidly than in the inertial range. The regime $k > k_d$ is known as the *dissipation range*. A simple scaling argument for k_d can be made by assuming that the spectrum follows the inertial scaling until k_d and then drops suddenly to zero because of viscous dissipation. In reality the transition between the two regimes is more gradual, but this simple model predicts k_d quite accurately. First we assume,

$$\begin{aligned} E(k) &= C_K \epsilon^{2/3} k^{-5/3}, \qquad k_i < k < k_d, \\ E(k) &= 0, \qquad k > k_d. \end{aligned} \tag{3.40}$$

Substituting (3.29), and integrating between k_i and k_d we find,

$$k_d \sim \left(\frac{\epsilon^{1/4}}{\nu^{3/4}} \right). \tag{3.41}$$

The inverse $l_d = 1/k_d$ is known as the *Kolmogorov scale*, the scale at which dissipation becomes important.

$$l_d \sim \left(\frac{\nu^{3/4}}{\epsilon^{1/4}} \right) \tag{3.42}$$

Integral scale

At the small wavenumber end of the spectrum, the important lengthscale is l_i, the integral scale, the scale of the energy-containing eddies. $l_i = 1/k_i$. We can evaluate l_i in terms of ϵ. Let us write,

$$U^2 = 2 \int_0^\infty E(k) \mathrm{d}k \tag{3.43}$$

and substituting for $E(k)$ from (3.36),

$$U^2 \sim 2 \int_0^\infty C_K \epsilon^{2/3} k^{-5/3} \mathrm{d}k \sim 3 C_K \epsilon^{2/3} k_i^{-2/3}. \tag{3.44}$$

Then,

$$k_i \sim \frac{\epsilon}{U^3} \tag{3.45}$$

so that $l_i \sim U^3/\epsilon$. Then the ratio of maximum and minimum dynamically active scales,

$$\frac{l_i}{l_d} = \frac{k_d}{k_i} \sim \frac{U^3}{\epsilon^{3/4} \nu^{3/4}} \sim \left(\frac{U l_i}{\nu} \right)^{3/4} \sim Re_{l_i}^{3/4}. \tag{3.46}$$

where Re_{l_i} is the *integral Reynolds number*. Hence in K41 the inertial range spans a range of scales growing as the (3/4)th power of the integral Reynolds number. It follows that if we want to describe such a flow accurately in a numerical simulation on a uniform grid, the minimum number of points per integral scale is $N \sim Re_{l_i}^{9/4}$. One consequence is that the storage requirements of numerical simulations scale as $Re_{l_i}^{9/4}$. Since the time step has usually to be taken proportional to the spatial mesh, the total computational work needed to integrate the equations for a fixed number of large eddy turnover times grows as $Re_{l_i}^3$. This shows that progress in achieving high Re_{l_i} simulations is very slow.

Taylor microscale

A third length scale often used to characterize turbulence is the *Taylor microscale*,

$$\lambda = \left(\frac{\langle u_i^2 \rangle}{\langle |\nabla u_i|^2 \rangle} \right)^{1/2} = \left(\frac{U^2 \nu}{\epsilon} \right)^{1/2}. \tag{3.47}$$

The Taylor microscale is the characteristic spatial scale of the velocity gradients. Using λ, an alternative Reynolds number can be defined,

$$Re_\lambda = \frac{U\lambda}{\nu} = \frac{U^2}{\nu^{1/2}\epsilon^{1/2}}, \tag{3.48}$$

where $Re_\lambda \sim Re_{l_i}^{1/2} \sim l_i/\lambda$.

Ozmidov scale

In geophysical flows 3D turbulence can be a reasonable approximation at scales small enough that buoyancy and rotation effects can be neglected. Stratification becomes important at scales smaller than rotation and it is therefore more important in setting the upper scales at which 3D arguments hold. Stratification affects turbulence when the Froude number $Fr = U/(NH) < 1$, where U is a typical velocity scale, and H a typical vertical length scale of the motion. For large Fr, the kinetic energy of the motion is much larger than the potential energy changes involved in making vertical excursions of order H. For small Fr, the stratification suppresses the vertical motion because a substantial fraction of kinetic energy must be converted to potential energy when a parcel moves in the vertical.

We can define a characteristic scale l_B at which overturning is suppressed by the buoyancy stratification as follows. The velocity associated with a particular length scale l in high Reynolds number isotropic 3D turbulence scales like,

$$u^2 \sim \epsilon^{2/3} k^{-2/3} \qquad \Longleftrightarrow \qquad u \sim (l\epsilon)^{1/3}. \tag{3.49}$$

Vertical motion at length scale l will be suppressed by the stratification when the local Froude number $Fr_l = 1$. If we define the length scale at which this suppression

occurs as l_B then,

$$\frac{u_B}{Nl_B} = \frac{(l_B\epsilon)^{1/3}}{Nl_B} = 1 \qquad \Longrightarrow \qquad l_B = \left(\frac{\epsilon}{N^3}\right)^{1/2} \tag{3.50}$$

where l_B is known as the *Ozmidov scale*.

In stratified geophysical flows, we have a scenario in which a regime transition occurs at l_B.

- $l < l_B$: Fully 3D, isotropic turbulence. In this regime stratification can be neglected, and an inertial range may exist, if $l_d << l_B$, i.e. $\epsilon/(\nu N^2) >> 1$.
- $l > l_B$: Stratification influenced regime. In this regime ϵ is no longer constant with wave number, since some kinetic energy is lost through conversion to potential energy. 3D turbulence is replaced by motion controlled by the buoyancy stratification: either internal waves, or a quasi-2-dimensional turbulence, often described as "pancake turbulence", characterized by strong vortical motions in decoupled horizontal layers.

(For more on stratified turbulence, see Lesieur Ch XIII, Metais and Herring, 1989: Numerical simulations of freely evolving turbulence in stably stratified fluids. *J. Fluid Mech.*, **239**. Fincham, Maxworthy and Spedding, 1996: Energy dissipation and vortex structure in freely-decaying stratified grid turbulence. *Dyn. Atmos. Oceans*, **23**, 155-169.)

Kolmogorov in physical space

Kolmogorov formulated his theory in physical space, making predictions for S_p, the longitudinal velocity structure function of order p,

$$\delta v_r = \left[\mathbf{u}(\mathbf{x}+\mathbf{r},t) - \mathbf{u}(\mathbf{x},t)\right].\frac{\mathbf{r}}{r} \tag{3.51}$$

$$S_p = \langle |\delta v_r|^p \rangle. \tag{3.52}$$

For homogeneous isotropic turbulence the structure function depends only on the magnitude of $\mathbf{r}$, i.e. $S_p = S_p(r)$. Under the assumptions described above, i.e. that at a scale r, S_p depends only on the energy flux ϵ, and the scale r, dimensional analysis can be sued to predict that,

$$S_p(r,t) = C_p(\epsilon r)^{p/3} \tag{3.53}$$

where C_p is a constant. In particular $S_2 \sim (\epsilon r)^{2/3}$. The second order structure function is related to the energy spectrum for an isotropic homogeneous field,

$$\begin{aligned} S_2 = \langle(\delta v_r)^2\rangle &= \langle(u_{//}(\mathbf{x}+\mathbf{r},t) - u_{//}(\mathbf{x},t))^2\rangle \\ &= 2\langle u_{//}(\mathbf{x}+\mathbf{r},t)u_{//}(\mathbf{x},t)\rangle + 2\langle|u_{//}(\mathbf{x},t)|^2\rangle \\ &= 2\int(1-e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}})\phi(\mathbf{k},t)\mathrm{d}\mathbf{k} \\ &= 4\int_0^\infty E(k,t)\left(1-\frac{\sin(kr)}{kr}\right)\mathrm{d}k. \end{aligned} \tag{3.54}$$

If we substitute for $E(k)$ from the Kolmogorov spectrum, and assume this applies from $k \gg r^{-1}$ then,

$$\langle(\delta v_r)^2\rangle \sim C_K(\epsilon r)^{2/3}. \tag{3.55}$$

Hence the Kolmogorov $k^{-5/3}$ spectrum is consistent with the second order structure function of the form $r^{2/3}$. (Note that S_2 is only finite if $E(k,t)$ has the form k^{-n} where $1 < n < 3$.)

3.2 Intermittency in isotropic 3D turbulence

[This chapter is a synthesis of material taken from Frisch (Turbulence, Cambridge University Press, 1995) and Salmon (Lectures on Geophysical Fluid Dynamics, Oxford University Press, 1998)]

Kolmogorov's 1941 model for the energy spectrum of isotropic homogeneous 3D turbulence assumes that there is an intermediate region in wavenumber space, the *inertial range*, where neither the forcing nor the viscosity are explicitly important. This simple assumption paves the road to determining the shape of the energy spectrum. Apart from forcing and dissipation, there are only two other dimensional parameters in the Navier-Stokes equations: the energy flux ϵ and the local wavenumber k. Thus the energy spectrum must be a function of these two parameters only. Dimensional consistency is all that is required to get an expression for the energy spectrum,

$$E(k) \approx C_K\epsilon^{2/3}k^{-5/3}. \tag{3.56}$$

The model assumes that the spectra at any particular k depends only on spectrally local quantities: the possibility for long-range interactions is ignored. In this chapter we discuss whether the assumption of locality is satisfied in real flows, and we examine some of the developments in the theory of 3D turbulence beyond Kolmogorov's seminal work.

Landau and the lack of universality in turbulence

In a famous footnote in his book on fluid dynamics, L.D. Landau noted an important inconsistency in K41 and objected to its universality. This led to a revision of the theory, but most people feel that it also destroyed the hope that there can be an exact theory. Landau's objection is neither the only, nor the most serious objection to K41. However it has helped the scientific community to better appreciate the enormous assumptions underlying Kolmogorov's theory.

Landau's remark appeared in a footnote in the 1944 edition of his book on *Fluid mechanics*, but in later editions found its way into the main text. Here is the full text of the remark, as it appears on page 140 of the second edition of the English translation of the book. The only changes are the substitution of Landau's notation, with the notation used in these notes.

One further general remark should be made. It might be thought that the possibility exist in principle of obtaining a universal formula, applicable to any turbulent flow, which should give $S_2(r)$ for all r that are small compared to r_0. In fact, however, there can be no such formula, as we see from the following argument. The instantaneous value of $(\delta v(r))^2$ might in principle be expressed as a universal function of the energy dissipation ϵ at the instant considered. When we average these expressions, however, an important part will be played by the manner of variation of ϵ over times of the order of periods of the large eddies (with size $\sim r_0$), and this variation is different for different flows. The result of the averaging therefore cannot be universal.

Kraichnan (1974) gave an illuminating reformulation of Landau's footnote remark. The essence of Landau's objection is that K41 cannot apply to a collection of flows with different dissipation rates ϵ. First consider two completely separate flows. The first flow is vigorously stirred so that ϵ_1 is large. The second flow is weakly stirred so that ϵ_2 is small. If both flows are fully turbulent, then according to K41,

$$E_1(k) = C_K \epsilon_1^{2/3} k^{-5/3}, \qquad \text{and} \qquad E_2(k) = C_K \epsilon_2^{2/3} k^{-5/3}. \tag{3.57}$$

Next consider a system composed of these two separate flows. If the flows occupy equal volumes, then the dissipation and the energy spectrum of the composite system are given by,

$$\epsilon = \frac{1}{2}\left(\epsilon_1 + \epsilon_2\right), \qquad \text{and} \qquad E(k) = \frac{1}{2}\left(E_1(k) + E_2(k)\right). \tag{3.58}$$

Thus for the composite system,

$$E(k) \neq C_K \epsilon^{2/3} k^{-5/3}. \tag{3.59}$$

That is, the composite system does not obey K41, essentially because the average of a two-thirds power is not equal to the two-thirds power of the average.

So far there seems to be no problem, because the composite flow is not a single flow, and hence there is no reason why K41 should apply to it. But suppose that the subscripts 1 and 2 do not refer to two flows, but to two large regions of the same flow with locally different dissipation rates. We conclude that K41 fails in cases where the dissipation rate ϵ, averaged over length scales characteristic of the inertial range, fluctuates.

Intermittency

Frisch in chapter 8 of his book on *Turbulence* shows two examples of irregular signals. The signal in Figure 8.1 is *self-similar*, *i.e.* successive enlargements of the signal have the same general aspect, regardless of where the magnification window is positioned. The signal in Figure 8.2 is *intermittent*, *i.e.* it displays activity during only a fraction of the time, which decreases with the scale under consideration. Enlargements of different sections of the signal produce completely different results, depending on whether the window is positioned on an active or passive period. When dealing with intermittent signals, the smaller the window, the more carefully it must be positioned to produce a nontrivial function.

The model of Kolmogorov relied on the assumption that turbulent signals are self-similar. Landau pointed out that, if dissipation is intermittent, then the model K41 had to be reconsidered. Laboratory experiments showed that Landau's remark was right on the spot: dissipation signals are strongly intermittent. In this section, we will discuss the theoretical arguments that have been proposed to reconcile Kolmogorov and Landau.

Definition

The notion of intermittency can be quantified for homogeneous isotropic random functions $v(\boldsymbol{x})$. Consider the structure function of order p,

$$S_p(r) = \langle\, |v(\boldsymbol{x}+\boldsymbol{r}) - v(\boldsymbol{x})|^p \,\rangle. \tag{3.60}$$

The structure function $S_p(r)$ depends only on the magnitude of $\boldsymbol{r}$ ($r \equiv |\boldsymbol{r}|$), because we assume that the turbulence, and therefore $v(\boldsymbol{x})$, is isotropic and homogeneous. We say that the random function $v(\boldsymbol{x})$ is intermittent at small scales if the kurtosis,

$$K(r) = \frac{S_4(r)}{S_2(r)^2} = \frac{\langle\, |v(\boldsymbol{x}+\boldsymbol{r}) - v(\boldsymbol{x})|^4 \,\rangle}{\langle\, |v(\boldsymbol{x}+\boldsymbol{r}) - v(\boldsymbol{x})|^2 \,\rangle^2} \tag{3.61}$$

grows without bound as the separation scale r decreases.

By our definition, neither Gaussian, nor self-similar signals are intermittent, because their kurtosis is independent of r. In the Gaussian case, this is because the difference of two Gaussian variables is also a Gaussian variable, and Gaussian variable have a kurtosis of 3. In the self-similar case the proof is straightforward as well. A random variable is self-similar if it posses a unique scaling exponent h, such that,

$$v(\boldsymbol{x} + \lambda \mathrm{r}) - v(\boldsymbol{x}) \overset{\text{law}}{=} \lambda^h \left(v(\boldsymbol{x} + \lambda \mathrm{r}) - v(\boldsymbol{x}) \right), \qquad \forall \lambda \in \Re, \tag{3.62}$$

for all $\boldsymbol{x}$, and all increments $\boldsymbol{r}$ and $\lambda \boldsymbol{r}$. It is easy to show that, for any $\lambda > 0$, when $\boldsymbol{r}$ is changed into $\lambda \boldsymbol{r}$ in (3.61), both the numerator and the denominator are multiplied by λ^h, leaving the flatness unchanged.

Is turbulence self-similar or intermittent? Visual inspection of turbulent velocity signals suggests that it is self-similar. If, however, velocity increments are computed over small enough distances, intermittent features show up. Intermittency becomes conspicuous only when the spatial separation $\boldsymbol{r}$ is comparable to, or smaller than, the Kolmogorov dissipation scale. Intermittency is thus a characteristic of the dissipation range.

The β model

The beta-model is a schematic model that illustrates the remark of Landau and suggests how to correct K41. We follow the derivation of Salmon in his book *Lectures on Geophysical Fluid Dynamics*.

Consider a turbulent flow stirred at some large scale r_0. The energy is transferred to smaller spatial scales via the nonlinear terms in the momentum equations. We suppose that this transfer happens in a series of cascade steps from scale r_0 to $r_1 = r_0/2$, from scale r_1 to $r_2 = r_1/2$, and so on (the factor of 1/2 is chosen for convenience; any other factor works as well). The n-th cascade step corresponds to an eddy size,

$$r_n = \frac{r_0}{2^n} \equiv k_n^{-1}. \tag{3.63}$$

We also define δv_n as the characteristic velocity change across eddies of size r_n; ϵ_n as the rate at which energy passes through the n-th cascade step; and

$$E_n = \int_{k_n}^{k_n+1} E(k)\ \mathrm{d}k, \tag{3.64}$$

as the energy, per unit volume, contained in eddies of size r_n.

We assume that the cascade can proceed in one of two ways, corresponding to Kolmogorov's and Landau's models of turbulence. In the first case, the eddies created at each scale fill the whole space uniformly, *i.e.* turbulence is self-similar. In the second case, eddies fill only a fraction β of the available space, and are correspondingly

stronger because they cascade the total energy flux in a smaller area. For $\beta < 1$ turbulence is intermittent in the sense proposed by Landau. K41 is recovered by setting $\beta = 1$.

For a generic $\beta \leq 1$, the total energy in eddies of size r_n is the energy within the eddies themselves, δv_n^2, times the fraction β^n of the total volume occupied by the eddies,

$$E_n \sim \beta^n \delta v_n^2. \tag{3.65}$$

This energy moves through the n-th cascade step in an eddy *turnover time*,

$$\tau_n \sim \frac{r_n}{\delta v_n} = \frac{1}{k_n \delta v_n}. \tag{3.66}$$

The rate at which energy passes through the n-th cascade step is then,

$$\epsilon_n \sim \frac{E_n}{\tau_n} \sim \beta^n \delta v_n^3 k_n. \tag{3.67}$$

If the turbulence is stationary, ϵ_n must be independent of n (otherwise energy would pile up at some intermediate wavenumber), that is,

$$\epsilon_n = \epsilon. \tag{3.68}$$

Combining (3.65), (3.67, and (3.68) we obtain,

$$E_n \sim \epsilon^{2/3} k_n^{-2/3} \beta^{n/3}. \tag{3.69}$$

The relationship between E_n and the spectrum, that is the kinetic energy density per unit wavenumber, follows from the definition in (3.64),

$$E_n = \int_{k_n}^{k_n+1} E(k)\ \mathrm{d}k = \int_{k_n}^{k_n+1} k\, E(k)\ \mathrm{d}\ln(k) = k_n E(k_n) \ln\left(\frac{k_{n+1}}{k_n}\right) \sim k_n E(k_n) \tag{3.70}$$

Plugging into (3.69),

$$E(k_n) \sim \epsilon^{2/3} k_n^{-5/3} \beta^{n/3}. \tag{3.71}$$

According to (3.71), the energy spectrum at wavenumber k_n is smaller than that predicted by K41, by a factor $\beta^{n/3}$ (and $\beta < 1$ if the cascade is not space filling). Physically this happens because eddies are more energetic, when they are not space filling, and their residency time is shortened at each cascade step.

Let us introduce a new parameter h as,

$$\beta = \frac{1}{2^h} \qquad h \geq 0. \tag{3.72}$$

We can now show that the intermittency of turbulence increases with h. Let us write,

$$\beta^n = \frac{1}{(2^h)^n} = \left(\frac{r_n}{r_0}\right)^h = \left(\frac{k_0}{k_n}\right)^h. \tag{3.73}$$

Using the definition of h, the spectrum in (3.71) becomes,

$$E(k_n) \sim k_0^{h/3}\epsilon^{2/3}k_n^{-(5+h)/3}. \tag{3.74}$$

The spectrum (3.74) reduces to K41 in the case of a space-filling cascade ($\beta = 1, h = 0$). However for intermittent ($h > 0$) turbulence, the spectrum falls off more steeply.

Observations support the prediction of K41 (with $h = 0$) for the spectrum, but suggest increasing disagreement with K41 as higher order moments are considered. Thus Kolomogorov's prediction for the spectrum remains valid, but Landau's remark plays an important role as well.

As an example of statistics of order higher than 2, we consider the structure functions defined in (3.60). If r^{-1} lies within the inertial range, we have shown that, by dimensional analysis, K41 predicts that,

$$S_p(r) = C_p(\epsilon r)^{p/3}, \tag{3.75}$$

and that the kurtosis is independent of r,

$$K(r) = \frac{S_4(r)}{(S_2(r))^2} = \frac{C_4}{(C_2)^2}. \tag{3.76}$$

However, observations suggest that $K(r)$ increases with r^{-1}. Since the kurtosis is a measure of intermittency, observations show a spatial intermittency that increases with decreasing eddy size. This contradicts K41, and suggests that eddies of decreasing size are confined to a decreasing fraction of fluid volume.

The β model accounts for the decreasing size of volume occupied by eddies at small scales. This results in a different scaling for the structure functions,

$$S_p\left(r_n\right) \sim \beta^n \delta v_n^p. \tag{3.77}$$

Hence, using (3.67) and $r_n = 1/k_n$, the β model predicts,

$$S_p\left(r_n\right) \sim \beta^{n(1-p/3)}\epsilon^{p/3}\left(\frac{r_n}{r_o}\right)^{p/3}, \tag{3.78}$$

or,

$$S_p\left(r_n\right) \sim C_p\epsilon^{p/3}\left(\frac{r_n}{r_o}\right)^{h(1-p/3)+p/3}. \tag{3.79}$$

We can now compute the kurtosis,

$$K(r) \sim \left(\frac{r_n}{r_o}\right)^{-h}. \tag{3.80}$$

The β model predicts that turbulence is intermittent, because $K(r)$ is an increasing function of r^{-1}.

It appears as if the hunt for the true model of homogeneous isotropic turbulence is still open. Observations support the predictions of K41 for the statistics of the second moment, but deviations are observed for $p > 2$. In particular $K(r)$ displays intermittency. The β model, with $h > 0$, predicts intermittency, but it also predicts that $S_2(r)$ should show departures from K41: these departures are not observed. In the next section we will show that a combination of both models seem to be required in order to explain the observations.

Multifractal models

In the literature, it is common to express the scalings of turbulent flows by computing the *exponents of structure functions* ζ_p, defined as,

$$S_p(r) \propto \left(\frac{r}{r_0}\right)^{\zeta_p}, \tag{3.81}$$

for a range of different p (not necessarily integers). K41 predicts that $\zeta_p = p/3$, while the β model has $\zeta_p = h(1 - p/3) + p/3$.

Observations suggest that $\zeta_p = p/3$ for $p \leq 3$, but deviations from this linear scaling appear at higher moments. A natural extension of the β model that produces a double scaling for different p is to introduce *bifractality*. Assume that there are two families of eddies both embedded in the volume of fluid. One family at each cascade step fills a fraction β_1 of the available space, while the other family fills a different fraction β_2. Together with β_1 and β_2, we also define h_1 and h_2 as in (3.72).

Following the same steps described for the β model, we can obtain the scalings for the structure functions,

$$S_p(r) = \mu_1 \left(\frac{r}{r_0}\right)^{h_1(1-p/3)+p/3} + \mu_2 \left(\frac{r}{r_0}\right)^{h_2(1-p/3)+p/3}, \tag{3.82}$$

where μ_1 and μ_2 are order unity constants.

Thus all the structure functions comprise the superposition of two power laws. In the inertial range, when $r \ll r_0$, the power law with the smallest exponent will dominate. We thus obtain,

$$S_p(r) \propto \left(\frac{r}{r_0}\right)^{\zeta_p}, \qquad \zeta_p = \min\left(h_1(1 - p/3) + p/3, h_2(1 - p/3) + p/3\right). \tag{3.83}$$

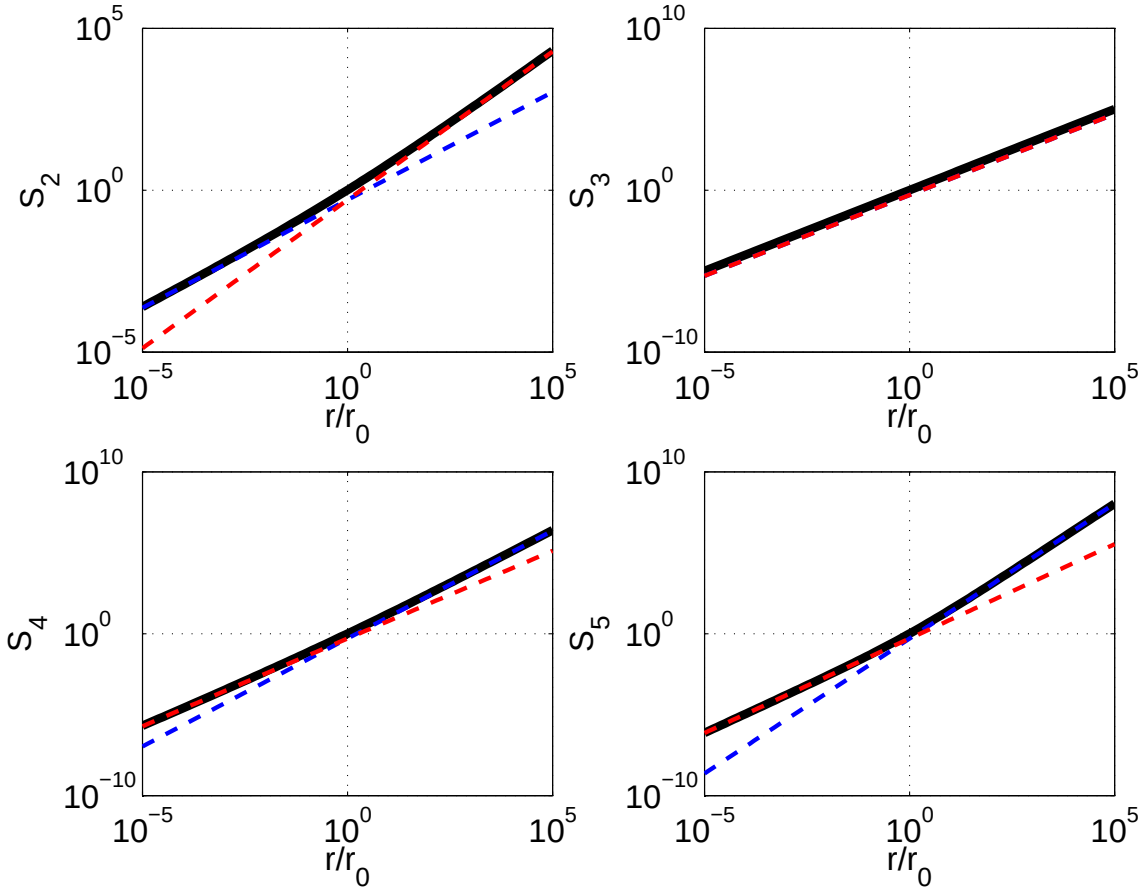

Figure 3.2: Structure functions of order 2, 3, 4, and 5 as a function of r/r_0, as given by eq. (3.82) with $h_1 = 0$ and $h_2 = .75$. The dark lines are the full S_p, the red and blue lines are the separate contributions of the two terms in the right-hand side of eq. (3.82)

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Depending on the value of the exponent p, the first or the second scaling dominates. This is reminiscent of the multiple scaling behavior found in observations (Figure 3.2);

As an illustration of what is called bifractality, let us take a mixture of K41 turbulence ($h_1 = 0$) and β model turbulence ($h_2 > 0$). We obtain,

$$\zeta_p = \begin{cases} p/3 & 0 \leq p \leq 3 \\ p/3 + h_2(1 - p/3) & p \geq 3. \end{cases} \tag{3.84}$$

Observe that with this choice the second (and third) moments follow the scaling of K41, but the fourth moment displays intermittency. The transition happens at $p = 3$, a point known as a *phase transition* (Fig. 3.2).

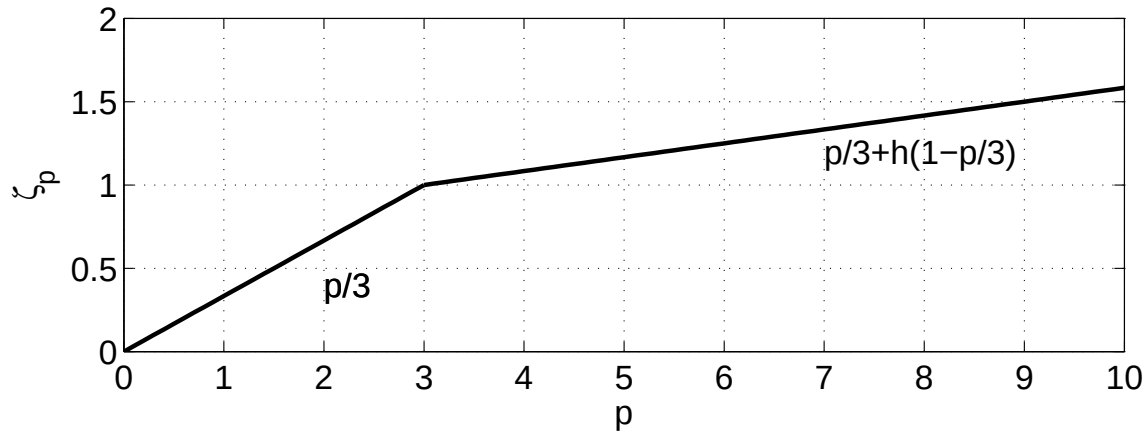

Figure 3.3: Multifractal behavior of the scaling exponents of the structure function given in eq. (3.82)
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In real world turbulence there are more than two scaling exponents h_1 and h_2. The curve ζ_p is neither a single power law (monofractal model), or the superposition of two power laws (bifractal model). Observations suggest that ζ_p behaves exactly as in the K41 theory for $p \leq 3$, but it does not follow a power law for larger p. *Multifractal models*, with multiple scaling exponents h_i, can be easily derived as extensions of the bifractal model. These models can be tuned to match the exponents ζ_p obtained from data.

Coherent structures

Half a century after Kolmogorov's work on the statistical theory of three dimensional turbulence, we still wonder how his work can be reconciled with Leonardo's half a millennium old drawings of eddy motion in the study for the elimination of rapids in the river Arno. Indeed, Kolmogorov's work on turbulence, ignores any structure which may be present in the flow.

In the first lecture, we pointed out that many turbulent flows are known to possess *coherent structures*. Their rediscovery by Crow and Champagne (1971) and Brown and Roshko (1974) has led to questioning the relevance of the traditional statistical theory of turbulence. The accepted paradigm is that, as far as the inertial-range properties are concerned, coherent structures do not matter if they are confined to the large scales of the flow. But is this really the case? And is there a fully developed inertial range in geophysical turbulence, where inhomogeneities and coherent structures do not appear?

A nice discussion of the dichotomy between the spectral description of turbulence and

the description based on coherent patterns in real space can be found in the paper by Armi and Flament (Journal of Geophysical Research, 90, 1985).

3.3 Passive tracer spectra

For a passive scalar which obeys an equation of the form ,

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \kappa \nabla^2 \theta, \tag{3.85}$$

we can write an equation for the variance $\langle \theta^2 \rangle$,

$$\frac{\partial \langle \theta^2 \rangle}{\partial t} + \nabla \cdot \langle \mathbf{u} \theta^2 \rangle = -\kappa \langle |\nabla \theta|^2 \rangle. \tag{3.86}$$

We assumed without loss of generality that $\langle \theta \rangle = 0$. Under the assumption that the tracer statistics are homogeneous and isotropic, we can write an equation for the spectrum $P(k)$ of this variance, analogous to (3.28),

$$2\kappa k^2 P(k) = T(k) + F(k), \tag{3.87}$$

where $T(k)$ is the nonlinear transfer of tracer variance, and $F(k)$ is an external source of tracer variance. Two of the results derived for the kinetic energy spectrum carry over to the tracer spectrum problem. (1) The dissipation of variance χ must equal the total injection of variance $\int_0^\infty F(k) \mathrm{d}k$. (2) At wavenumbers far from the injection scale and dissipation scale, variance is fluxed at a constant rate χ (set by the injection rate). Using these two results, we can derive the form of the spectrum $P(k)$. Notice however that there is a major difference between the kinetic energy and the tracer problems. In the tracer inertial range χ and k are not the only relevant parameters, since the tracer field is subject to stirring by the flow. The flow parameters (e.g., ϵ) also influence the tracer field.

We can derive the shape of the tracer spectrum in the range of wavenumbers where both tracer and momentum dissipation can be neglected. Once again we assume that forcing is confined to large scales. In the so-called *inertial-convective range* the fluxes of kinetic energy and tracer variance must be constant, if a statistically steady sate is to be achieved. Thus we can state, in analogy to Obukhov's argument for kinetic energy, that the tracer flux is given by the available variance at wavenumber k divided by the eddy turnover timescale,

$$\chi \sim \frac{kP(k)}{\tau}. \tag{3.88}$$

Assuming that eddy stirring is dominated by local interactions we can write that $\tau = [k^3 E(k)]^{-1/2}$. But χ is a constant and therefore we have,

$$P(k) \sim \chi k^{-5/2} E(k)^{-1/2} \tag{3.89}$$

Substituting for $E(K)$ from K41 we have,

$$P(k) = \beta\chi\epsilon^{-1/3}k^{-5/3} \tag{3.90}$$

where β is some universal constant. The tracer spectrum in the inertial-convective range has the same slope as the kinetic energy spectrum and is known as the Obukhov-Corrsin spectrum.

Length scales

The kinetic energy spectrum becomes influenced by viscosity at a wavenumber k_d such that $Re \sim 1$. In order to estimate the Reynolds number at a particular lengthscale, we need a scaling for the velocity field. Using K41 we have,

$$\langle \delta v_r^2 \rangle \sim (\epsilon r)^{2/3} \qquad \Longrightarrow \qquad v_r \sim (\epsilon r)^{1/3}, \tag{3.91}$$

where v_r is an order of magnitude estimate of the velocity at a lengthscale r. Then

$$Re_r \sim \frac{v_r r}{\nu}. \tag{3.92}$$

Setting $Re_r \sim 1$, we find that viscosity becomes important at the scale $1/r = k_d \sim (\epsilon/\nu^3)^{1/4}$, the Kolmogorov scale.

By analogy with the kinetic energy spectrum, the passive tracer spectrum becomes influenced by diffusion at a wavenumber k_c, where the Peclet number ~ 1. We have two different scenarios, depending on whether the wavenumber k_c is smaller or larger than the Kolmogorov wavenumber k_d.

If the Prandtl number $Pr = \nu/\kappa < 1$, then the dissipation scale k_c occurs within the inertial range ($k_c < k_d$). Plugging $v_r \sim (\epsilon r)^{1/3}$ in the definition of the Peclet number,

$$Pe_r \sim \frac{v_r r}{\kappa}, \tag{3.93}$$

we find that $Pe_r \sim 1$ is achieved at a wavenumber $1/r = k_c \sim (\epsilon/\kappa^3)^{1/4} = Pr^{3/4}k_d$.

However, if diffusion becomes important at wavenumbers larger than viscosity does (i.e. $Pr > 1$), k_c does not lie within the inertial range, so we cannot use the inertial range scaling to obtain v_r; if the energy spectrum $E(k)$ drops off more rapidly than k^{-3}, then $(\delta v_r)^2$ cannot be calculated from (3.54). In this range the velocity spectrum drops off exponentially to zero. Thus at scales k shorter than the Klolmogorov scale, the tracer is not stirred by eddies with scale k because such eddies do not exist. At these scales the trcaer is stirred by the smallest scales present in the flow, i.e. by

eddies at the Kolmogorv scale. For these eddies $v_r \sim (\epsilon/k_d)^{1/3} \sim \nu k_d$. Smaller scale features feel this as a "large-scale" flow. Then the local Peclet number at a scale r is,

$$Pe_r = \frac{v_r r}{\kappa} = \frac{\nu k_d r}{\kappa}. \tag{3.94}$$

By definition $Pe_r \sim 1$ when $r = 1/k_c$, the wavenumber at which diffusion becomes important. Thus,

$$k_c \sim \frac{\nu}{\kappa} k_d. \tag{3.95}$$

Depending on the relative length of the viscous and dissipative cutoff scales, the passive tracer tracer spectrum has several different subranges. For $k_i << k$, and $k << k_d$ and $k << k_c$, neither κ nor ν are important. This is the *inertial-convective range* considered above. If $k << k_d$, but $k > k_c$ (for $Pr < 1$) then κ is important, but not ν: the spectrum is in an *inertial-diffusive range*. If $k << k_c$, but $k > k_d$ (for $Pr > 1$), then ν is important but not κ: the spectrum is in an *viscous-convective range*. Finally for $k > k_d$ and $k > k_c$, the spectrum is in a *viscous-diffusive range*. We consider the spectrum in each of these subranges separately.

Inertial-diffusive range

In the inertial diffusive range the flux of variance is no longer constant with k, since diffusion is acting to reduce it. Instead, from (3.87),

$$T(k) = -\frac{d\Pi}{dk} = 2\kappa k^2 P(k). \tag{3.96}$$

The flux $\Pi(k)$ is not a constant in k in this range. Using Obukhov's argument we can also write,

$$\Pi(k) = \frac{kP(k)}{[k^3 E(k)]^{-1/2}}. \tag{3.97}$$

Inertial range scaling for the energy still applies, so we can use K41 to express $E(k)$ and we find that,

$$P(k) \sim \Pi(k) k^{-5/2} E(k)^{-1/2} = \beta \epsilon^{-1/3} k^{-5/3} \Pi(k). \tag{3.98}$$

Substituting for $P(k)$ in (3.96) we have,

$$\frac{d\Pi}{dk} = -2\beta\kappa\epsilon^{-1/3} k^{1/3} \Pi(k). \tag{3.99}$$

Solving for $\Pi(k)$ we get,

$$\Pi(k) = \chi \exp\left[-\frac{3}{2}\beta\kappa\epsilon^{-1/3} k^{4/3}\right]. \tag{3.100}$$

If we substitute back into (3.98) we find,

$$P(k) = \beta\epsilon^{-1/3}k^{-5/3}\chi \exp\left[-\frac{3}{2}\beta\left(\frac{k}{k_c}\right)^{4/3}\right] \tag{3.101}$$

where $k_c = (\epsilon/\kappa^3)^{1/4}$. Hence the spectrum of tracer variance behaves exponentially for $k > k_d$ when $Pr < 1$. This spectrum is not valid far into the inertial-diffusive subrange because it assumes $\Pi(k)$ varies only slowly with k. (An alternative theory of Batchelor et al. (1959) gives a $k^{-17/3}$ spectrum. Neither form of the spectrum has been verified.)

Viscous-convective subrange

For $Pr > 1$ and $k > k_d$, but $k < k_c$, the flux of variance $\Pi(k)$ is constant: $\Pi(k) = \chi$. κ is not important, but ν is. The energy field drops off rapidly for $k > k_d$. Hence the scalar perturbations experience a shear corresponding to that at a scale k_d, $v_{kd}k_d = (\epsilon/\nu)^{1/2}$. At $k > k_d$ this shear appears like a smooth large-scale flow. $P(k)$ must satisfy,

$$\chi = \frac{kP(k)}{[k_d^3 E(k_d)]^{-1/2}}. \tag{3.102}$$

Plugging the expression for the Kolmogorov wavenumber k_d,

$$P(k) = C_B\chi k^{-1}\left(\frac{\epsilon}{\nu}\right)^{-1/2}. \tag{3.103}$$

This is known as the **Batchelor spectrum**, and C_B is the Batchelor constant.

There is experimental evidence for the Batchelor spectrum. Gibson and Schwarz (JFM, 1963) observed the Batchelor spectrum for temperature and salinity in laboratory measurements in water, and the approximate behavior for temperature spectrum is also suggested by field measurements of Grant et al. (JFM, 1968), Oakey and Elliott (JPO, 1982) and others. There is however a wide scatter in the predicted values of the universal constant C_B. The practical importance of these spectral expressions lies in the fact that all scalar fluctuations and scalar dissipation are effectively determined by scales from the Batchelor range. The dissipation rates in turn determine the mixing coefficients for scalars which are critical to understand small-scale physics of the oceans and large scale circulation and global climate. The knowledge of spatial power spectra of temperature fluctuations at small scales is also needed in treating problems of sound and light propagation in water.

Further reading: Lesieur, Ch V, VI; Tennekes and Lumley, Ch 8; Frisch, Ch 5, 6, 7, 8.