Turbulent motions in the Atmosphere and Oceans

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Course description

The course will present the phenomena, theory, and modeling of turbulence in the Earth's oceans and atmosphere. The scope will range from centimeter to planetary scale motions. The regimes of turbulence will include homogeneous isotropic three dimensional turbulence, convection, boundary layer turbulence, internal waves, two dimensional turbulence, quasi-geostrophic turbulence, and planetary scale motions in the ocean and atmosphere. Prerequisites: the mathematics and physics required for admission to the graduate curriculum in the EAPS department, or consent of the instructor.

Course requirements

Class attendance and discussion, weekly homework assignments.

Reference texts

Andrews, Holton, and Leovy, "Middle atmosphere dynamics"
Frisch, "Turbulence: the legacy of Kolmogorov"
Lesieur, "Turbulence in Fluids", 3rd revised edition
McComb, "The physics of turbulence"
Saffman, "Vortex dynamics"
Salmon, "Lectures on geophysical fluid dynamics"
Tennekes and Lumley, "A first course in Turbulence"
Whitham, "Linear and nonlinear waves"

Chapter 4

Transport of passive and active tracers in turbulent flows

A property of turbulence is to greatly enhance transport of tracers. For example, a dissolved sugar molecule takes years to diffuse across a coffee cup, relying only on molecular agitation (actually on that time scale the coffee will surely evaporate). With a spoon, the coffee drinker can create eddies that transport dissolved sugar throughout the cup in less than a second. This enhanced transport is generally described as an *eddy diffusivity*.

The concept of eddy diffusivity is often justified by appealing to an analogy between turbulent eddies and molecular diffusion. The argument goes that turbulent eddies move tracer parcels in erratic motions, much alike bombardment by molecular agitation in Brownian motion, and thus the action of turbulence may be represented as an enhanced diffusion. This concept was formalized by Prandtl in a body of work known as mixing length theory. Despite the name, however, there is little theory in the argument. In this lecture we will introduce the concept of mixing length theory and then we will consider simple systems where it is possible to explain very explicitly the assumptions behind mixing length arguments and to illustrate the situations in which the analogy fails.

4.1 Mixing length theory

In Lecture 1 we derived the averaged momentum and buoyancy equations in the Boussinesq approximation. (Equations for other conserved tracers have the same form as the equation for buoyancy.) These equations include average velocities, pressure and buoyancy as well as eddy stresses and fluxes, i.e. correlations between fluctuations,

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = \bar{b}\delta_{i,3} - \frac{1}{\rho_0}\frac{\partial}{\partial x_j}\left(\overline{p}\delta_{i.j} - \rho_0 \nu \frac{\partial \overline{u}_i}{\partial x_j} + \rho_0 \overline{u'_j u'_i}\right), \tag{4.1}$$

$$\frac{\partial \overline{u}_j}{\partial x_j} = 0, \tag{4.2}$$

$$\frac{\partial \bar{b}}{\partial t} + \overline{u_j}\frac{\partial \bar{b}}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(-\kappa \frac{\partial \bar{b}}{\partial x_j} + \overline{b' u'_j}\right). \tag{4.3}$$

When dealing with atmospheric or oceanic flows the averaging is generally assumed to be a space and time filter on the scales of the large scale circulation, while perturbation quantities u'_i and b' include all small-scale processes. Somewhat inconsistently it is also assumed that the averaging operator satisfies the all the usual properties for ensemble averages, like $\overline{\bar{b}b'} = 0$.

The topic of this chapter is the parameterization of the perturbation terms, which amounts to finding a way to express the small-scale quantities $\overline{u'_j u'_i}$ (the Reynolds stresses) and $\overline{b' u'_j}$ (the Reynolds fluxes) in terms of large-scale quantities $(\bar{u}_i, \bar{b})$.

Eddy viscosity and eddy diffusivity

The simplest parameterization of the turbulent fluxes/stresses employs an eddy viscosity assumption,

$$\overline{u'_j u'_i} = -\nu_T \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{4.4}$$

$$\overline{u'_j b'} = -\kappa_T \frac{\partial \bar{b}}{\partial x_j} \tag{4.5}$$

This is a first order closure. Models vary in the complexity of the system used to specify ν_T and κ_T, the eddy viscosity and eddy diffusivity. ν_T, κ_T can be specified directly in terms of the large-scale quantities of the flow. This is the approach followed by Prandtl's in the mixing length model. More elaborate models have been derived since, where the ν_T and κ_T are specified in terms of small-scale quantities for which extra prognostic equations are required. We will see some of these higher order closures at the end of the class when we discuss boundary layer models.

Mixing length model

Consider a parcel in a 2-D shear flow $(\bar{u}(y), 0, 0)$, initially at at some position y. If the parcel moves due to turbulent motion, up to a position $(x + \delta x, y + \delta y)$, and it conserves momentum, then it has a momentum deficit compared to the parcels around it,

$$u' = [\bar{u}(y+\delta y) - \bar{u}(y)] + \delta u \approx \delta y \frac{\partial \bar{u}}{\partial y} + \delta u, \tag{4.6}$$

$$v' = -\delta v, \tag{4.7}$$

where δu and δv are the random velocity fluctuations that displaced the particle. We truncated the Taylor expression of $\bar{u}$ to the first terms under the assumption that,

$$\delta y \ll \frac{\partial \bar{u}/\partial y}{\partial^2 \bar{u}/\partial y^2}. \tag{4.8}$$

If we further assume that the statistics of turbulent fluctuations are homogeneous and isotropic,

$$\overline{u'v'} = -\overline{\delta y \delta v}\frac{\partial \bar{u}}{\partial y}. \tag{4.9}$$

Introducing the *mixing length* ℓ - the distance at which δv and δy become uncorrelated - we can write,

$$\overline{\delta y \delta v} = -c\ \ell \left(\overline{\delta v^2}\right)^{1/2}, \tag{4.10}$$

where c is a constant. We then have,

$$\overline{u'v'} = -\nu_T \frac{\partial \bar{u}}{\partial y} = -c\ \ell \left(\overline{\delta v^2}\right)^{1/2} \frac{\partial \bar{u}}{\partial y}, \tag{4.11}$$

where ν_T is the eddy viscosity,

$$\nu_T = c\ \ell \left(\overline{\delta v^2}\right)^{1/2}. \tag{4.12}$$

Under the isotropy assumption $\overline{\delta v^2} = \overline{\delta u^2} = \overline{\delta w^2}$, we can write this as

$$\nu_T = c\ \ell\ \sqrt{q} \tag{4.13}$$

where $q/2$ is the eddy kinetic energy, the kinetic energy of the turbulent fluctuations. Eq.(4.13) could also be obtained on dimensional grounds, by assuming the turbulent motion is characterized by a single velocity scale $\sqrt{q}$, and a single lengthscale ℓ.

Analogous arguments can be repeated for tracers and one obtains,

$$\overline{u'b'} = -\kappa_T \frac{\partial \bar{b}}{\partial y}, \tag{4.14}$$

where κ_T is the eddy diffusivity,

$$\kappa_T = c_T\ \ell\ \sqrt{q}, \tag{4.15}$$

and c_T is a constant, possibly different from c. The mixing length argument does not tell us anything about the ratio $\nu_T/\kappa_T = c/c_T$. It is often assumed that the *turbulent Prandtl number* $\nu_T/\kappa_T \approx 1$, and that turbulent transport of buoyancy and momentum are equally efficient. However, this is one of the adjustable parameters of a turbulence parameterization. If we assume constant eddy viscosity and diffusivity, we are assuming that a single mixing length ℓ and eddy kinetic energy $q = \overline{v'^2}$ characterize the flow at all points in space and time. Obviously this cannot be true for the ocean and atmosphere, where turbulence is highly inhomogeneous. A way out of this apparent inconsistency is to assume that turbulence is homogeneous on scales smaller than the large-scale flow and thus we can apply mixing length theory.

An issue of concern is that eddy mixing length theory should not be used for non-conserved quantities. If we assume that the average is carried on distances so short that pressure effects do not change momentum much, then we can apply eddy mixing length theory to momentum. However this is often done in numerical and theoretical models whose resolution is too coarse for this to be true. The issue of momentum parameterization is one that should be kept in mind

4.2 Transport of passive tracers

We now revisit the parameterization of eddy transport in a more systematic way. Once again we use a Reynolds decomposition of variables into mean and eddy components. We depart from the advection-diffusion equation for a generic passive tracer of concentration c,

$$c_t + \boldsymbol{u} \cdot \nabla c = \kappa \nabla^2 c, \tag{4.16}$$

where κ is the molecular diffusivity and $\boldsymbol{u}$ is an incompressible ($\nabla \cdot \boldsymbol{u} = 0$) velocity field. The velocity field is given in this problem, i.e. we do not write a momentum equation to solve for the velocity field. The equation for the mean tracer concentration is,

$$\bar{c}_t + \bar{\boldsymbol{u}} \cdot \nabla \bar{c} + \nabla \cdot \overline{\boldsymbol{u}'c'} = \kappa \nabla^2 \bar{c}. \tag{4.17}$$

The equation for the fluctuations is,

$$c'_t + \bar{\boldsymbol{u}} \cdot \nabla c' + \nabla \cdot \left[\boldsymbol{u}'c' - \overline{\boldsymbol{u}'c'}\right] - \kappa \nabla^2 c' = -\boldsymbol{u}' \cdot \nabla \bar{c}. \tag{4.18}$$

We can see that advective distortion of the mean gradient, $\nabla \bar{c}$, generates fluctuations, c'.

4.2.1 Effective diffusivity and the multi-scale method

If $c' = 0$ at $t = 0$ then, c' and $\nabla \bar{c}$ will be linearly related. This is obvious by inspection of eq. (4.18). The equation for the tracer fluctuations is linear in c' for a prescribed velocity field and it is forced by $\nabla\bar{c}$: doubling the forcing doubles the tracer fluctuations. It follows that the eddy flux $\overline{\boldsymbol{u}'c'}$ will also be linearly related to the mean gradient $\nabla\bar{c}$. These simple considerations, together with an assumption of scale separation between the eddies and the mean, can be used to extract a surprising amount of information.

The scale separation approach is based on the assumption that the advected tracer is weakly inhomogeneous on a scale L much greater than the scale l_0 of the background turbulent fluctuations. The goal is then to derive equations for the evolution of the coarse-grained averaged tracer on the length scale L, and on a time scale T large compared with the time scale $t_0 \sim l_0/u_0$, characteristic of the energy containing eddies of the turbulence. We may then choose an intermediate scale λ, *e.g.* $\lambda = (l_0 L)^{1/2}$, and an intermediate time τ, *e.g.* $\tau = (t_0 T)^{1/2}$, satisfying,

$$l_0 \ll \lambda \ll L, \qquad t_0 \ll \tau \ll T, \tag{4.19}$$

and think of the overbar average as a “local average” over a cube of side λ, and a time of order τ. Averaged quantities will vary only on larger scales of order L and longer times of order T. The replacement of ensemble and volume/time averages is possible only if the turbulence is homogeneous and stationary on the small spatial and temporal scales, so that an ergodic assumptions can be made.

The multi-scale method that we use here was first introduced by Papanicolaou and Pirroneau (1981). The scale separation assumption suggests that a perturbation expansion can be done in terms of the small parameter $\epsilon \equiv l_0/L$. Suppose now that $c(\boldsymbol{x}, 0)$ is slowly varying so that,

$$c(\boldsymbol{x}, 0) = \mathcal{C}_0(\epsilon\boldsymbol{x}). \tag{4.20}$$

Eq. (4.16), together with the initial condition in eq. (4.20) suggests a multiple scale analysis with the slow variables,

$$\boldsymbol{X} = \epsilon\boldsymbol{x}, \boldsymbol{X}_2 = \epsilon^2\boldsymbol{x}, \qquad T = \epsilon t, T_2 = \epsilon^2 t. \tag{4.21}$$

The solution of eq. (4.16) then takes the form,

$$c(\boldsymbol{x}, t; \epsilon) = C_0(\boldsymbol{X}, T) + \epsilon\, C_1(\boldsymbol{x}, t; \boldsymbol{X}, T) + \epsilon^2 C_2(\boldsymbol{x}, t; \boldsymbol{X}, T) + \dots . \tag{4.22}$$

The quantity of interest is the large-scale, long-time, averaged field $\bar{c} = C_0(\boldsymbol{X}, T) + O(\epsilon)$. Its evolution is obtained by usual asymptotic methods. Substituting the expansion (4.22) into the advection diffusion equation (4.16), one obtains a series of equations order by order in ϵ.

The advecting velocity field must be expanded as well. We will assume that the velocity field is composed of a mean flow varying on the slow variables only $\boldsymbol{U}(\boldsymbol{X}, T)$ and a turbulent perturbations $\boldsymbol{u}'$,

$$\boldsymbol{u} \equiv \boldsymbol{U}(\boldsymbol{X}, T) + \boldsymbol{u}'(\boldsymbol{x}, t; \boldsymbol{X}, T). \tag{4.23}$$

We do not assume that the mean flow is of small amplitude compared to the turbulent flow as it is often done in the literature of eddy mean flow interactions

Let's write the series of equations order by order in ϵ. At fist order we have,

$$\boldsymbol{O(\epsilon^0)}: \qquad C_{0t} + (\boldsymbol{U} + \boldsymbol{u}') \cdot \nabla_{\boldsymbol{x}} C_0 - \kappa \nabla_{\boldsymbol{x}}^2 C_0 = 0. \tag{4.24}$$

The solution to this equation, satisfying the assumption that the initial tracer concentration is smooth, has the general form,

$$C_0 = \mathcal{C}_0(\boldsymbol{X}, T, \boldsymbol{X}_2, T_2). \tag{4.25}$$

$$\boldsymbol{O(\epsilon)}: \qquad C_{1t} + (\boldsymbol{u}' + \boldsymbol{U}) \cdot \nabla_{\boldsymbol{x}} C_1 - \kappa \nabla_{\boldsymbol{x}}^2 C_1 = -C_{0T} - (\boldsymbol{u}' + \boldsymbol{U}) \cdot \nabla_{\boldsymbol{X}} C_0. \tag{4.26}$$

Averaging over the small and fast scales scales, we have,

$$C_{0T} + \boldsymbol{U} \cdot \nabla_{\boldsymbol{X}} C_0 = 0, \tag{4.27}$$

from which it follows that,

$$C_{1t} + (\boldsymbol{u}' + \boldsymbol{U}) \cdot \nabla_{\boldsymbol{x}} C_1 - \kappa \nabla_{\boldsymbol{x}}^2 C_1 = -\boldsymbol{u}' \cdot \nabla_{\boldsymbol{X}} C_0. \tag{4.28}$$

Solutions to this problem can be written in the form $C_1 = -(\boldsymbol{\xi} \cdot \nabla) C_0 + \mathcal{C}_1(\boldsymbol{X}, T)$, with $\boldsymbol{\xi}(\boldsymbol{x}, t)$ satisfying the equation,

$$\boldsymbol{\xi}_t + (\boldsymbol{U} + \boldsymbol{u}') \cdot \nabla_{\boldsymbol{x}} \boldsymbol{\xi} - \kappa \nabla_{\boldsymbol{x}}^2 \boldsymbol{\xi} = \boldsymbol{u}'. \tag{4.29}$$

This equation resembles the equation for a particle displacement, except for the presence of the molecular diffusive term. This difference is extremely important, because molecular diffusion is ultimately the only process that can mix the tracer. Any theory of diffusion which neglects molecular processes must be taken with suspicion. The terms $\mathcal{C}_1(\boldsymbol{X}, T)$ represents a small correction to the initial tracer concentration.

$$\boldsymbol{O(\epsilon^2)}: \qquad C_{2t} + (\boldsymbol{u}' + \boldsymbol{U}) \cdot \nabla_{\boldsymbol{x}} C_2 - \kappa \nabla_{\boldsymbol{x}}^2 C_2 = \tag{4.30}$$

$$-C_{1T} - (\boldsymbol{u}' + \boldsymbol{U}) \cdot \nabla_{\boldsymbol{X}} C_1 + 2\kappa \nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{X}} C_1 + \tag{4.31}$$

$$\kappa \nabla_{\boldsymbol{X}}^2 C_0 - C_{0T_2} - (\boldsymbol{U} + \boldsymbol{u}') \cdot \nabla_{\boldsymbol{X}_2} C_1. \tag{4.32}$$

By taking the large scale and long time average of this equation, we obtain the solvability condition,

$$C_{0T_2} + \boldsymbol{U} \cdot \nabla_{\boldsymbol{X}_2} C_0 + C_{1T} + \boldsymbol{U} \cdot \nabla_{\boldsymbol{X}} C_1 \;\; = \;\; \kappa \nabla_{\boldsymbol{X}}^2 C_0 + \overline{(\boldsymbol{u}' \cdot \nabla_{\boldsymbol{X}}) \, \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{X}} C_0}. \tag{4.33}$$

The rhs represents the diffusion of tracer concentration by molecular and turbulent processes. This is best seen if we write,

$$\kappa \nabla^2_{\boldsymbol{X}} C_0 + \overline{(\boldsymbol{u}' \cdot \nabla_{\boldsymbol{X}}) \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{X}} C_0} = \nabla_{\boldsymbol{X}} \cdot \left[\kappa \nabla_{\boldsymbol{X}} C_0 + \overline{\boldsymbol{u}' \boldsymbol{\xi}} \cdot \nabla_{\boldsymbol{X}} C_0 \right] \tag{4.34}$$

$$= \nabla_{\boldsymbol{X}} \cdot [\boldsymbol{D} \nabla_{\boldsymbol{X}} C_0], \tag{4.35}$$

where the tensor $\boldsymbol{D}$,

$$D_{ij} = \kappa \delta_{ij} + \overline{u'_i \xi_j}, \tag{4.36}$$

is the effective diffusivity tensor.

Summing the solvability conditions at $O(\epsilon)$ and $O(\epsilon^2)$ we obtain the evolution equation for the mean tracer concentration,

$$\bar{c}_t + \bar{\boldsymbol{u}} \cdot \nabla \bar{c} = \partial_{x_i} [D_{ij} \partial_{x_j} \bar{c}]. \tag{4.37}$$

The mean tracer concentration is given by $\bar{c} = \mathcal{C}_0 + \epsilon \mathcal{C}_1$ and derivatives include variations on the slow space and time of first seond order. The effective diffusivity $\boldsymbol{D}$ is obtained by solving eq. (4.29) and computing the correlations between $\boldsymbol{\xi}$ and the velocity fluctuations $\boldsymbol{u}'$.

The main result of the multiple scale analysis is that there is a relationship between the eddy flux $\overline{\boldsymbol{u}' c'}$ and the mean tracer gradient $\nabla \bar{c}$,

$$\overline{u'_i c'} = -D_{ij} \partial_{x_j} \bar{c}. \tag{4.38}$$

If the turbulence is isotropic, then D_{ij} is likewise isotropic, *i.e* $D_{ij} = D \delta_{ij}$, and eq. (4.38) becomes,

$$\overline{\boldsymbol{u}' c'} = -D \ \nabla \bar{c}. \tag{4.39}$$

a linear diffusive relationship, in which D can be interpreted as the eddy diffusivity, in perfect analogy with the molecular diffusivity. Note however that the analogy holds only under the stringent assumptions of scale separation between fluctuations and mean, and of homogeneity, isotropy, and stationarity of the turbulent fluctuations.

If the turbulence is not isotropic, it is tempting to regard D_{ij} as an anisotropic diffusion tensor. This interpretation is misleading as we are about to show. The diffusivity tensor can be decomposed into its symmetric and antisymmetric components,

$$D = D^s + D^a, \tag{4.40}$$

where the symmetric component is,

$$D^s_{ij} \equiv \frac{1}{2} \left(D_{ij} + D_{ji} \right), \tag{4.41}$$

and the antisymmetric component is,

$$D_{ij}^a \equiv \frac{1}{2}\left(D_{ij} - D_{ji}\right). \tag{4.42}$$

Using the equation for $\boldsymbol{\xi}$, we obtain useful expressions for the diffusive and skew components of the diffusivity,

$$\begin{aligned} \overline{u_i'\xi_j} &= \overline{\left(\xi_{i,t} + u_k'\xi_{i,k} + U_k\xi_{i,k} - \kappa\xi_{i,kk}\right)\xi_j} \\ &= \kappa\overline{\xi_{i,k}\xi_{j,k}} + \frac{1}{2}\left[\overline{\xi_j\xi_{i,t} - \xi_i\xi_{j,t}} + \overline{\xi_j\left(u_k + U_k\right)\xi_{i,k} - \xi_i\left(u_k + U_k\right)\xi_{j,k}}\right] \end{aligned} \tag{4.43}$$

The first term on the rhs is symmetric and represents the diffusive component. The second term is asymmetric and is the skew component.

The diffusive component of the diffusivity represents processes that tend to remove mean tracer variance. This can be seen by inspecting the tracer variance equation,

$$\begin{aligned} \partial_t\bar{c}^2 + \nabla\cdot\left(\bar{\boldsymbol{u}}\bar{c}^2 + \boldsymbol{D}\nabla\bar{c}^2\right) &= -D_{ij}^s\partial_{x_i}\bar{c}\ \partial_{x_j}\bar{c} \\ &= -\kappa\overline{\left(\xi_{i,j}\ \partial_{x_i}\bar{c}\right)^2} \leq 0 \end{aligned} \tag{4.44}$$

The variance of the mean tracer is only dissipated by the symmetric component of the diffusivity tensor. The antisymmetric component moves variance around, but it does not dissipate it.

The role of the antisymmetric component of the diffusivity tensor is best explained if we write $\boldsymbol{D}^a$ in the form,

$$D_{ij}^a = \epsilon_{ijk}\Psi_k. \tag{4.45}$$

This is the generic form of an antisymmetric tensor in three dimensions, i.e. any antisymmetric tensor can be written in the form (4.45). The tracer flux associated to the antisymmetric component of the diffusivity is the so called *skew flux*,

$$\overline{\boldsymbol{u}'c'}^a = -\boldsymbol{\Psi}\times\nabla\bar{c}. \tag{4.46}$$

If turbulence is homogeneous on the large scale L, then D_{ij} is uniform in space, and therefore so are $\boldsymbol{D}^a$ and $\boldsymbol{\Psi}$. In this limit case, eq. (4.46) implies that,

$$\nabla\cdot\overline{\boldsymbol{u}'c'}^a = 0, \tag{4.47}$$

and the skew flux $\overline{\boldsymbol{u}'c'}^a$ makes no contribution to the mean field equation (4.37). This is because eq. (4.46) describes tracer transfer parallel to surfaces of constant $\bar{c}$. However, if the turbulence is inhomogeneous, then $\boldsymbol{D}^a$ will be a function of $\boldsymbol{X}$, and so,

$$\nabla\cdot\overline{\boldsymbol{u}'c'}^a = -\nabla\cdot\left(\boldsymbol{\Psi}\times\nabla\bar{c}\right) = -\nabla\times\boldsymbol{\Psi}\cdot\nabla\bar{c} = \bar{\boldsymbol{u}}_S\cdot\nabla\langle c\rangle, \tag{4.48}$$

where,

$$\bar{\boldsymbol{u}}_S = -\nabla \times \boldsymbol{\Psi}, \tag{4.49}$$

and this substituted in eq. (4.37) implies advection of $\bar{c}$ by a generalized Stokes drift $\bar{\boldsymbol{u}}_S$.

In summary then,

- The symmetric part of the diffusivity tensor corresponds to something like diffusive transport.
- The antisymmetric part, which is almost never zero, and it is in fact usually dominant for rotationally dominated waves, corresponds to an advective transport. As a result, the mean advecting velocity that appears in eq. (4.37) is not $\bar{\boldsymbol{u}}$, but the velocity $\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}_S$. This seems to be telling us that the Eulerian mean velocity $\langle \boldsymbol{u} \rangle$ is not the most natural choice of "mean" for this problem.

4.2.2 Relationship to Lagrangian description

The Lagrangian displacement of a partcile advected by the sum of the mean and turbulent flows is given by,

$$\frac{d\xi_i}{dt} = u_i(\boldsymbol{\xi}, t).$$

On the fast time scale, particles are only displaced by the short distance associated with the turbulent eddy scale. Thus we can solve for $\boldsymbol{\xi}$ by iteration,

$$\frac{d\xi_i}{dt} = u_i(\boldsymbol{x}, t) + \int_0^t u_j(\boldsymbol{x}, t')\partial_{x_j} u_j(\boldsymbol{x}, t)\ dt'$$

Thus the estimated mean Lagrangian motion is,

$$\begin{aligned}
\bar{u}_{Li} = \overline{\frac{d\xi_i}{dt}} &= \bar{u}_i(\boldsymbol{x}, t) + \partial_{x_j} \int_0^t \overline{u_j(\boldsymbol{x}, t')u_j(\boldsymbol{x}, t)}\ dt' \\
&= \bar{u}_i(\boldsymbol{x}, t) + \partial_{x_j} \int_0^t R_{ij}(\boldsymbol{x}, \tau)\ d\tau && (4.50) \\
&= \bar{u}_i(\boldsymbol{x}, t) + \bar{u}_{Si}(\boldsymbol{x}, t) + \partial_{x_j} \int_0^t \frac{1}{2}\left[R_{ij}(\boldsymbol{x}, \tau) + R_{ji}(\boldsymbol{x}, \tau)\right]\ d\tau && (4.51) \\
&= \bar{u}_i(\boldsymbol{x}, t) + \bar{u}_{Si}(\boldsymbol{x}, t) + \partial_{x_j} D^s_{ij}. && (4.52)
\end{aligned}$$

Notice that the mean Lagrangian velocity includes a drift term due to gradients of the symmetric diffusivity tensor in addition to the familiar mean Eulerian velocity and

Stokes drift contributions. This drift represents the tendency for particles to spread toward regions of high diffusivity.

Now consider the advection-diffusion equation for the mean tracer concentration that we derived in the previous section,

$$\partial_t \bar{c} + (\bar{u}_i + \bar{u}_{Si})\, \partial_{x_i} \bar{c} = \partial_{x_i} \left(D^s_{ij} \partial_{x_j} \bar{c} \right)$$

Rearranging terms we can write,

$$\partial_t \bar{c} + \partial_{x_i} \left[\left(\bar{u}_i + \bar{u}_{Si} + \partial_{x_j} D^s_{ij} \right) \bar{c} \right] = \partial_{x_i x_j} \left(D^s_{ij} \bar{c} \right)$$

This equation is known as the forward Chapman-Kolmogorov equation and describes the evolution of the mean tracer concentration is a Lagrangian framework. Indeed the advection is by the mean Lagrangian flow in this form.

A puzzling aspect of this result is that the mean Lagrangian is typically nondivergent because of the drift term $\partial_{x_j} D^s_{ij}$.. This can be understood if we consider that the nondivergent condition in lagrangian coordinates is,

$$\frac{\partial(x+\xi, y+\eta, z+\zeta)}{\partial(x, y, z)} = 1 \tag{4.53}$$

so that,

$$\frac{\partial \xi_i}{\partial x_i} + \frac{\partial(\xi, \eta)}{\partial(x, y)} + \frac{\partial(\xi, \zeta)}{\partial(x, z)} + \frac{\partial(\eta, \zeta)}{\partial(y, z)} + \frac{\partial(\xi, \eta, \zeta)}{\partial(x, y, z)} = 0. \tag{4.54}$$

If we take the time derivative and the long time average of all terms, we find that the first term represents the divergence of the mean Lagrangian velocity, while all the other terms represent the degree to which that velocity is divergent.

We could consider the backward problem, where the Lagrangian tags are the final particle positions at time t,

$$\begin{aligned}
\bar{u}_{Li} = \overline{\frac{d\xi_i}{dt}} &= \bar{u}_i(\boldsymbol{x}, t) - \partial_{x_j} \int_{-t}^{0} \overline{u_j(\boldsymbol{x}, t') u_j(\boldsymbol{x}, t)}\, dt' \\
&= \bar{u}_i(\boldsymbol{x}, t) - \partial_{x_j} \int_{-t}^{0} R_{ij}(\boldsymbol{x}, \tau)\, d\tau && (4.55) \\
&= \bar{u}_i(\boldsymbol{x}, t) + \bar{u}_{Si}(\boldsymbol{x}, t) + \partial_{x_j} \int_0^t \frac{1}{2} \left[R_{ij}(\boldsymbol{x}, \tau) - R_{ji}(\boldsymbol{x}, \tau) \right]\, d\tau && (4.56) \\
&= \bar{u}_i(\boldsymbol{x}, t) + \bar{u}_{Si}(\boldsymbol{x}, t) - \partial_{x_j} D^s_{ij}. && (4.57)
\end{aligned}$$

In this case the corresponding Lagrangian equation for the tracer concentration is the backward Chapman-Kolmogorov equation,

$$\partial_t \bar{c} + \left(\bar{u}_i + \bar{u}_{Si} - \partial_{x_j} D^s_{ij} \right) \partial_{x_i} \bar{c} = D^s_{ij} \partial_{x_i x_j} \bar{c}.$$

In this case the Lagrangian mean velocity is biased away from region of large symmetric diffusivities, because we are considering where particles are most likely to be coming from.

4.3 Transport of momentum

The above discussion is valid only for tracers that satisfy an advection diffusion equation. Is it possible to extend the argument to momentum and define an *effective viscosity*? This problem is discussed at length in the notes of Alan Plumb on "Eddy transport in the atmosphere and the ocean", and we follow closely that presentation.

Let us consider, for simplicity, a barotropic velocity field (that is a 2D system). The absolute vorticity $\zeta_a = f + v_x - u_y$ satisfies an advection-diffusion equation of the form,

$$\partial_t \zeta_a + \boldsymbol{u} \cdot \nabla \zeta_a = F_y - E_x. \tag{4.58}$$

where E and F represent friction or other forces. Momentum does not satisfy a conservation equation, because of nonlocal pressure terms. As we are about to show, we can use the conservation of absolute vorticity to make some sense of momentum transport.

Let's start with the zonal momentum equation,

$$u_t + u u_x + v\ (u_y - f) = -\frac{1}{\rho_0} p_x + F, \tag{4.59}$$

where F is the zonal acceleration due to friction or other forces. We now want to consider the zonal average of eq. (4.59). The zonal mean is defined as,

$$\bar{u}(y,t) \equiv \frac{1}{L} \int_{-L/2}^{L/2} a(x,y,t)\ \mathrm{d}x. \tag{4.60}$$

Note that,

$$\bar{u}_x = 0, \qquad \bar{v}_y = 0, \qquad \bar{p}_x = 0. \tag{4.61}$$

The zonal average of eq. (4.59) is then,

$$\bar{u}_t + \bar{v}\ (\bar{u}_y - f) = -\partial_y \overline{u'v'} + \bar{F}. \tag{4.62}$$

The eddy term is due to the northward flux of zonal momentum $\overline{u'v'}$. Because of pressure gradients, momentum is not conserved by eddy motions and we will see that there is no basis to expect a relationship of the form,

$$\overline{u'v'} = -D \partial_y \bar{u}. \tag{4.63}$$

But vorticity satisfies an advection diffusion equation and we can use the results on scalar turbulence derived in the previous section. We showed that for scalars, it is possible to relate the eddy fluxes to the mean tracer distribution. Thus we might expect that there is a relationship between absolute vorticity fluxes and the mean vorticity gradient. This relationship can be further used to relate the momentum fluxes to a mean gradient. Note in fact that,

$$\begin{aligned}\overline{v'\zeta'} &= \overline{v'v'_x} - \overline{v'u'_y} && (4.64)\\ &= -\partial_y \overline{u'v'}. && (4.65)\end{aligned}$$

Thus we can rewrite the zonal mean momentum equation as,

$$\bar{u}_t + \bar{v}\left(\bar{u}_y - f\right) = \overline{v'\zeta'} + \bar{F}. \tag{4.66}$$

So that, whereas in eq. (4.62) the eddies appear as an agency of momentum transport through the eddy momentum flux $\overline{u'v'}$, in eq. (4.66) they appear as an eastward body force, equal to the northward eddy flux of vorticity, acting on the mean.

The zonal mean vorticity equation reads,

$$\bar{\zeta}_t + \bar{v}\bar{\zeta}_y = \bar{F}_y - \partial_y \overline{v'\zeta'}, \tag{4.67}$$

and, by appealing to the arguments given in the previous section we can write,

$$\overline{v'\zeta'} = -D\partial_y \bar{\zeta}. \tag{4.68}$$

This relationship can used to derive an expression for the eddy momentum flux,

$$\partial_y \overline{u'v'} = D\partial_y \bar{\zeta} = D\left(\beta - \bar{u}_{yy}\right). \tag{4.69}$$

If $\beta = 0$ and D is uniform, then we can integrate this expression (assuming vanishing flux at the boundaries),

$$\overline{u'v'} = -D\ \partial_y \bar{u}. \tag{4.70}$$

But D cannot be uniform. For example the vorticity flux must vanish at the boundaries where $v' = 0$, so this result is a little suspect. Moreover, if β is nonzero, then the whole approach does not work and mean velocity gradients might not be important at all. The main conclusion is that turbulent fluctuations cannot be represented as an enhanced viscosity in the momentum equation. But not all is lost, because progress can be made by considering the vorticity flux, rather than the momentum flux.