Turbulent motions in the Atmosphere and Oceans

Instructors: Raffaele Ferrari and Glenn Flierl

Course description

The course will present the phenomena, theory, and modeling of turbulence in the Earth's oceans and atmosphere. The scope will range from centimeter to planetary scale motions. The regimes of turbulence will include homogeneous isotropic three dimensional turbulence, convection, boundary layer turbulence, internal waves, two dimensional turbulence, quasi-geostrophic turbulence, and planetary scale motions in the ocean and atmosphere. Prerequisites: the mathematics and physics required for admission to the graduate curriculum in the EAPS department, or consent of the instructor.

Course requirements

Class attendance and discussion, weekly homework assignments.

Reference texts

Andrews, Holton, and Leovy, "Middle atmosphere dynamics"
Frisch, "Turbulence: the legacy of Kolmogorov"
Lesieur, "Turbulence in Fluids", 3rd revised edition
McComb, "The physics of turbulence"
Saffman, "Vortex dynamics"
Salmon, "Lectures on geophysical fluid dynamics"
Tennekes and Lumley, "A first course in Turbulence"
Whitham, "Linear and nonlinear waves"

Chapter 5

Effective diffusivity of cellular flows

[Parts of this chapter follow Bill Young's notes on diffusion in cellular flows (http://www-pord.ucsd.edu/ wryoung/StirrMix/cellDiffChpt.pdf)]

In the previous section we dealt with passive scalar turbulence in a generic three-dimensional flow. Now we turn to the simpler case of passive tracers advected by incompressible two-dimensional cellular flows. Cellular flows are flows that are doubly periodic on a square domain. Here we will use cellular flows to illustrate the concept of effective diffusivity and the difference between its symmetric and antisymmetric components.

In two dimensions the velocity field can be obtained from a streamfunction $\psi(x, y)$ according to the definition $\boldsymbol{u} = (u, v) = (-\psi_y, \psi_x)$. The domain is a periodic array of square cells, each of size l, so the streamfunction has the periodicity $\psi(x+ml, y+nl) = \psi(x, y)$, where m and n are integers. We use the notation,

$$\langle c \rangle \equiv \frac{1}{l^2} \int\int c \,\mathrm{d}x\mathrm{d}y, \tag{5.1}$$

to denote the average over an $l\times l$ square. We are assuming that the average over a cell vanishes, i.e. $\langle \boldsymbol{u} \rangle = 0$. The cell average $\langle \quad \rangle$ will play a role analogous to the large-scale, long-time average of the previous section. Thus we will be concerned with the "large-scale" transport of a passive tracer, where "large-scale" means a length which is much greater than the cell size l. In this simple configuration, a cell represents the scale of turbulent fluctuations, and the cell average is used to isolate the slowly varying part of the concentration.

As an illustrative example, start with the steady state advection-diffusion equation,

$$J(\psi, c) = \kappa \nabla^2 c, \tag{5.2}$$

where $J(a, b) = a_x b_y - a_y b_x$ is the Jacobian and κ is the molecular diffusivity.

If we release some tracer into a steady cellular flow does the blob spread diffusively? Without molecular diffusivity ($\kappa = 0$), the answer is clearly no. Each tracer particle will stay on its initial streamline, and if that streamline is closed then there can be no large-scale transport. But, with even very weak molecular diffusivity, molecules of tracer are not confined to streamlines and indeed there is an effective diffusivity using the Gx-trick (Young, 1999). That is, we suppose that a large scale uniform gradient $\boldsymbol{G}$ is externally imposed and we then proceed to calculate the flux $\boldsymbol{F}$ which is associated with $\boldsymbol{G}$. This procedure enables us to bypass the initial value problem and deal with a simpler *steady state* problem.

Suppose that the system is in a big box containing $N \times N$ cells, i.e. the box is a $Nl \times Nl$ square. On the wall at $x = 0$, we impose the boundary condition $c(0, y) = 0$ and on the wall at $x = Nl$, we impose $c(Nl, y) = GNl$. Further, suppose that there is no flux of c through the boundaries at $y = 0$ and $y = N$. If there is no advection ($\psi = 0$) then the solution of (5.2) with these boundary conditions is $c = Gx$, and the flux associated with this $\psi = 0$ solution is $F_0 = -\kappa G$.

The procedure of assuming a steady large scale gradient is used to bypass the initial value problem and deal with a simpler steady state problem in (5.2). The first step to find a solution is to make the simple substitution,

$$c = \boldsymbol{G} \cdot \boldsymbol{x} + c'(x, y), \tag{5.3}$$

which separates c into the large-scale uniform gradient and a periodic flow-induced perturbation c'. Throwing (5.3) into (5.2) one obtains,

$$J(\psi, c') - \kappa \nabla^2 c' = -\boldsymbol{G} \cdot \boldsymbol{u}. \tag{5.4}$$

Equation (5.4) is linear in c' and a general solution can be written in the form,

$$c' = -\boldsymbol{a} \cdot \mathbf{G}, \tag{5.5}$$

where $\boldsymbol{a} \equiv [a(x, y), b(x, y)]$ is determined by solving,

$$J(\psi, \boldsymbol{a}) - \kappa \nabla^2 \boldsymbol{a} = \boldsymbol{u}. \tag{5.6}$$

The eddy-flux averaged over the doubly-periodic domain $\mathbf{F} \equiv \langle \mathbf{u'c'} \rangle$ can now be written expressing c' as a linear combination of a and b,

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = -\begin{bmatrix} \langle ua \rangle & \langle ub \rangle \\ \langle va \rangle & \langle vb \rangle \end{bmatrix} \begin{pmatrix} G_x \\ G_y \end{pmatrix}. \tag{5.7}$$

The 2×2 matrix above is the effective diffusivity tensor $\mathbf{K}$. If $\mathbf{G}$ is not uniform, then (5.7) is the first term in an expansion of the form $F_i = -K_{ij}G_j + L_{ijk}G_{j,k} + \ldots$. Here we limit the analysis to the leading order effect contained in $\mathbf{K}$.

Equation (5.6) can now be used to estimate the terms in the rhs of (5.7). Multiplying both sides by $\boldsymbol{a}$ and integrating by parts we obtain,

$$
\begin{aligned}
\langle ua \rangle &= \kappa \langle \nabla a \cdot \nabla a \rangle, && (5.8)\\
\langle ub \rangle &= \langle \boldsymbol{u} a \cdot \nabla b \rangle + \kappa \langle \nabla a \cdot \nabla b \rangle, && (5.9)\\
\langle va \rangle &= \langle \boldsymbol{u} b \cdot \nabla a \rangle + \kappa \langle \nabla a \cdot \nabla b \rangle, && (5.10)\\
\langle vb \rangle &= \kappa \langle \nabla b \cdot \nabla b \rangle. && (5.11)
\end{aligned}
$$

In order to write in a compact form the effective diffusivity tensor $\mathbf{K}$, we introduce the streamfunction ψ, associated with the stirring field $(u', v') = (-\partial_y \psi, \partial_x \psi)$. We also separate the symmetric and antisymmetric parts of $\mathbf{K} = \mathbf{K^s} + \mathbf{K^a}$ and write,

$$
\mathbf{K^s} = \left[\begin{array}{cc} \kappa \langle \nabla a \cdot \nabla a \rangle & \kappa \langle \nabla a \cdot \nabla b \rangle \\ \kappa \langle \nabla a \cdot \nabla b \rangle & \kappa \langle \nabla b \cdot \nabla b \rangle \end{array} \right], \tag{5.12}
$$

and

$$
\mathbf{K^a} = \left[\begin{array}{cc} 0 & \langle \psi J(a,b) \rangle \\ -\langle \psi J(a,b) \rangle & 0 \end{array} \right]. \tag{5.13}
$$

The anti-symmetric part of $\mathbf{K}$ is equivalent to advection and represents a Stokes' drift. Let $\phi \equiv -\langle \psi J(a,b) \rangle$ and $\boldsymbol{u}_\phi \equiv (-\partial_y \phi, \partial_x \phi)$. In a slowly varying large scale field, the averaged concentration evolves according to,

$$
\partial_t \langle c \rangle + J(\psi + \phi, \langle c \rangle) = \nabla \cdot \mathbf{K^s} \, \nabla \langle \mathbf{c} \rangle. \tag{5.14}
$$

5.1 A simple example

Consider a two dimensional eddy field given by the streamfunction,

$$
\psi = \Psi \sin(kx) \sin(ky), \qquad \boldsymbol{u} = (u, v) = (-\psi_y, \psi_x), \tag{5.15}
$$

i.e. a periodic array of eddies of size $l \times l$, which alternate between rotating clockwise and anticlockwise ($k = 2\pi/l$). A picture of the flow pattern is shown in the first panel of Fig. 5.1. Instead of considering the initial value problem for the tracer advection-diffusion equation, we use the Gx-trick to obtain an expression for the eddy diffusivity in this flow.

Once again we make the substitution,,

$$
c = \boldsymbol{G} \cdot \boldsymbol{x} + c'(x, y), \tag{5.16}
$$

which separates c into the large-scale uniform gradient and a periodic flow-induced perturbation c', and we solve

$$
J(\psi, c') - \kappa \nabla^2 c' = -\boldsymbol{G} \cdot \boldsymbol{u}, \tag{5.17}
$$

with the ψ given in (5.15).

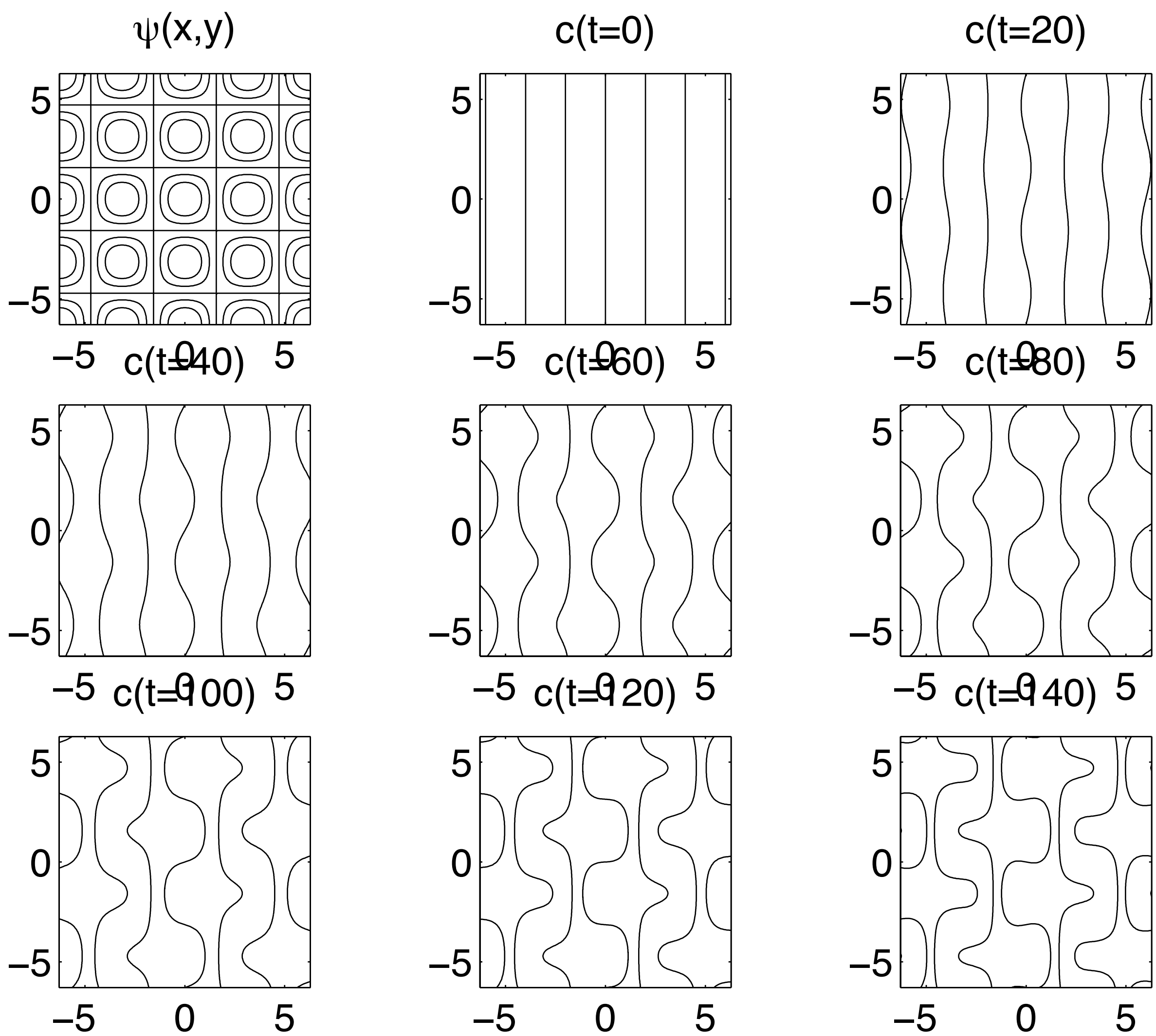

Figure 5.1: Tracer contours at different times for a tracer advected by the streamfunction $\Psi \sin(kx) \sin(ky)$.

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5.1.1 The small Peclet number limit

It is impossible to solve eq. (5.17) exactly. However we can make progress if we assume that diffusion dominates over advection at leading order, i.e.,

$$J(\psi, c') \sim \frac{\Psi \Delta C}{l^2} \ll \kappa \nabla^2 c' \sim \frac{\kappa \Delta C}{l^2}, \tag{5.18}$$

where ΔC is the characteristic scale of tracer fluctuation in the flow. The inequality in (5.18) is satisfied for small Peclet number $Pe \equiv \Psi/\kappa$. Thus we will consider the effective diffusivity in a small Peclet number limit.

The equation for $\boldsymbol{a}$ in (5.6), in the small Peclet number limit, reduces to,

$$-\kappa \nabla^2 \boldsymbol{a} = \boldsymbol{u}. \tag{5.19}$$

whose solutions can be easily found to be,

$$a = -\frac{\Psi}{2k\kappa} \sin(kx)\cos(ky), \qquad b = \frac{\Psi}{2k\kappa} \cos(kx)\sin(ky). \tag{5.20}$$

These expressions can be used to estimate all terms that appear in the effective diffusivity tensor,

$$\kappa \langle \nabla a \cdot \nabla a \rangle = \frac{1}{8}\frac{\Psi^2}{\kappa} = \frac{1}{8} Pe^2 \kappa, \tag{5.21}$$

$$\kappa \langle \nabla b \cdot \nabla b \rangle = \frac{1}{8}\frac{\Psi^2}{\kappa} = \frac{1}{8} Pe^2 \kappa, \tag{5.22}$$

$$\kappa \langle \nabla a \cdot \nabla b \rangle = 0, \tag{5.23}$$

$$\langle \psi J(a, b) \rangle \rangle = 0. \tag{5.24}$$

Thus, to leading order in the Peclet number, the effective diffusivity for this flow is given by,

$$\boldsymbol{K} = \frac{1}{8} Pe^2 \kappa \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.25}$$

In this example, the advecting field generates a symmetric diffusivity tensor, i.e. diffusion is enhanced by the advecting field but there is no skew flux.

5.2 An example with skew flux

In Fig. 5.2 we show an example of a flow which generates a skew component. This flow is composed of eddies all rotating clockwise,

$$\psi = \Psi \sin^2(kx)\sin^2(ky), \qquad \boldsymbol{u} = (u, v) = (-\psi_y, \psi_x), \tag{5.26}$$

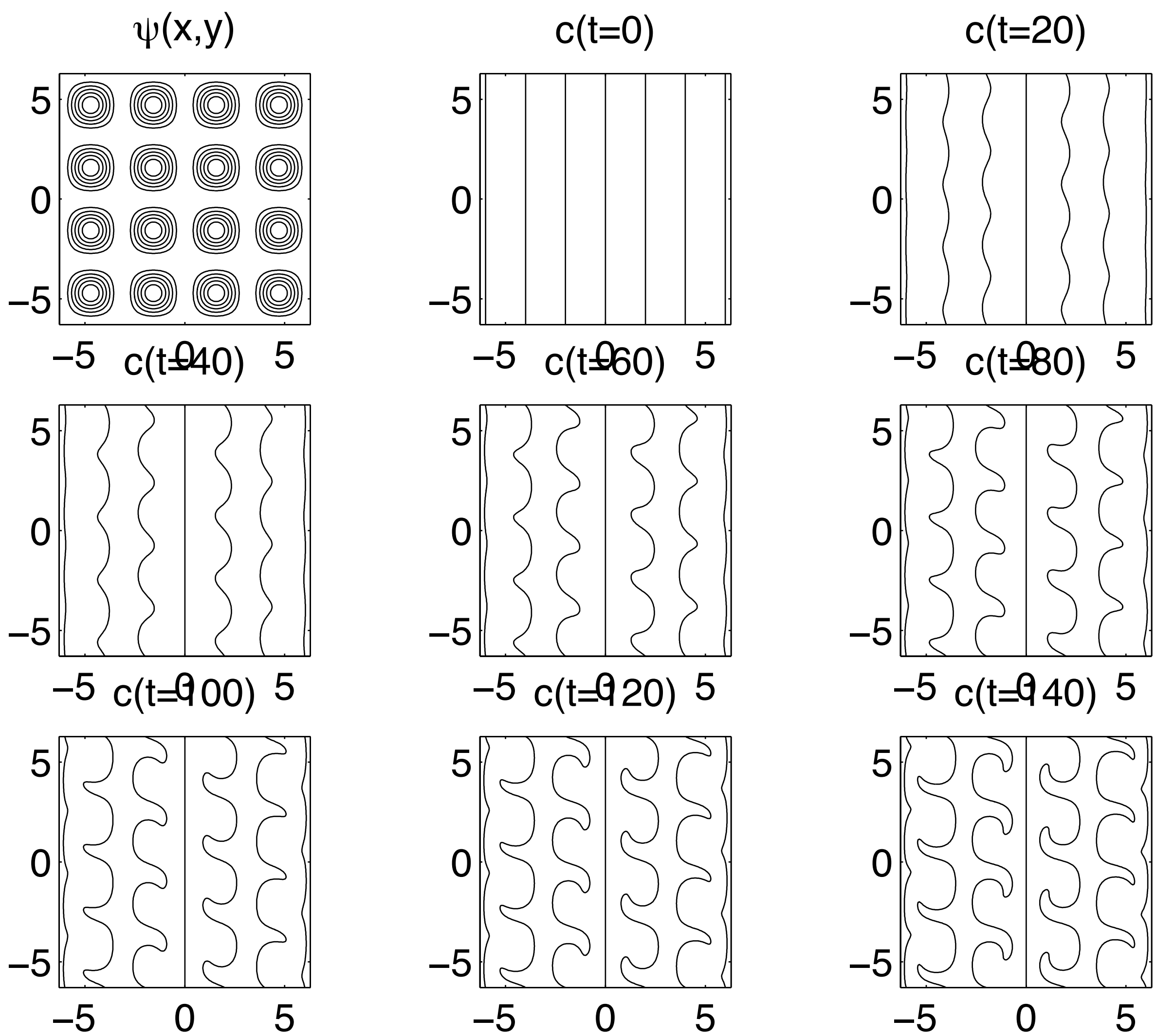

Figure 5.2: Tracer contours at different times for a tracer advected by the streamfunction $\Psi \sin^2(kx)\sin^2(ky)$.
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As a result we can see that the tracer is transported up and down in the vertical by the eddy field. This flux resembles advection by vertical jets.

We can work out the small Pe number expansion following the same steps described above. After a somewhat laborious algebra we find,

$$\boldsymbol{K} = \frac{1}{128} Pe^2 \kappa \begin{bmatrix} 5 - \frac{269}{1280} Pe^2 & Pe \\ -Pe & 5 - \frac{269}{1280} Pe^2 \end{bmatrix}. \tag{5.27}$$

In this problem the symmetric tensor is diffusive and isotropic. The novelty is that at order Pe^3 there is a skew flux. The skew flux is constant in space and therefore there is no Stokes drift associated with it. A Stokes drift would appear if we added a modulation of the cellular velocity pattern on a large scale, i.e.,

$$\psi = \Psi(\epsilon x, \epsilon y) \sin^2(kx) \sin^2(ky). \tag{5.28}$$

5.3 The effect of a mean flow

Consider now the cellular flow described above in the presence of a mean flow $(U_0, 0)$. That is consider dispersion in the flow,

$$\psi = \Psi \sin(kx) \sin(ky), \qquad \boldsymbol{u} = (u + U_0, v) = (-\psi_y + U_0, \psi_x). \tag{5.29}$$

In order to have a mean flow, we must assume that there is a forcing term in the tracer equation that maintains the large scale tracer gradient $\boldsymbol{G}$ against the action of advection.

5.3.1 The small Peclet number limit

Once again we consider the small Pe number limit. The leading order solution is identical to the problem discussed above. However if we press on, we obtain the correction to dispersion introduced by the presence of a mean flow. The solutions for the a and b fields are,

$$a = -\frac{Pe}{2k} \sin(kx)\cos(ky) - \frac{Pe^2}{16k} \sin(2kx) - \frac{Pe^2 U_0}{4\Psi k^2} \cos(kx)\cos(ky), \tag{5.30}$$

$$b = \frac{Pe}{2k} \cos(kx)\sin(ky) + \frac{Pe^2}{16k} \sin(2ky) + \frac{Pe^2 U_0}{4\Psi k^2} \sin(kx)\sin(ky). \tag{5.31}$$

The corresponding effective diffusivity tensor is,

$$\boldsymbol{K} = \frac{1}{8} Pe^2 \kappa \begin{bmatrix} 1 - \frac{1}{16} Pe^2 - \frac{U_0^2}{4\Psi^2 k^2} Pe^2 & 0 \\ 0 & 1 - \frac{1}{16} Pe^2 - \frac{U_0^2}{4\Psi^2 k^2} Pe^2 \end{bmatrix}. \tag{5.32}$$

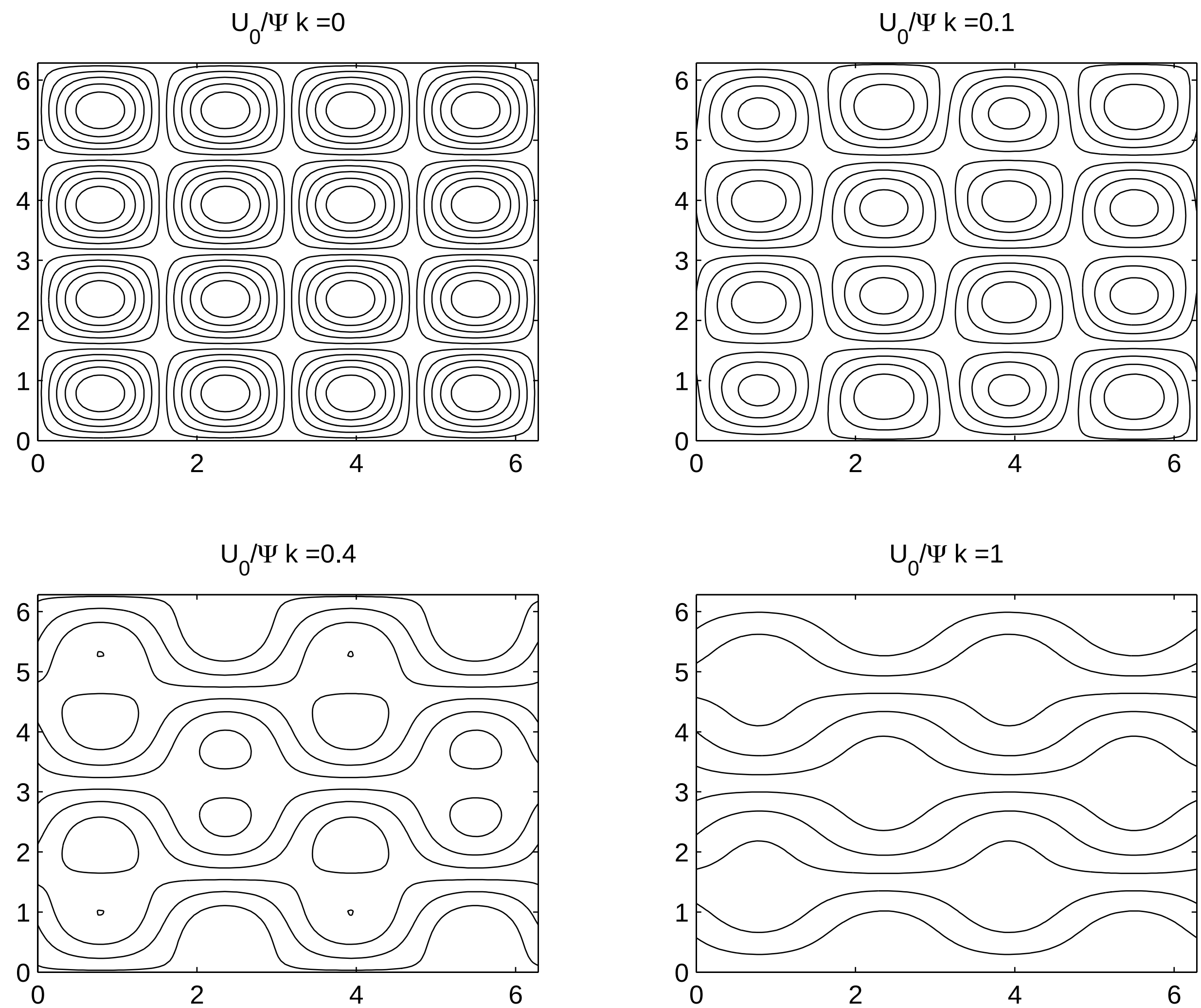

Figure 5.3: Contours of a cellular flow with the addition of a mean flow.

This solution shows that the presence of a mean flow suppresses dispersion in the flow. This effect appears at order Pe^4 as long as $U_0 = O(k\Psi)$. The explanation is simple: a mean flow reduces the regions of convergence where the cells create large gradients and enhance diffusivity as shown in figure 5.3.1.

5.3.2 The strong mean flow limit

If the mean flow is much larger than the eddy flow, we can solve the full transport problem, because to leading order,

$$U_0\boldsymbol{a}_x - \kappa\nabla^2\boldsymbol{a} = \boldsymbol{u}. \tag{5.33}$$

Solutions have the form,

$$a = -2\frac{\Psi k\kappa}{U_0^2 + 4\kappa^2 k^2}\sin(kx)\cos(ky) + \frac{\Psi U_0}{U_0^2 + 4\kappa^2 k^2}\cos(kx)\cos(ky), \quad (5.34)$$

$$b = 2\frac{\Psi k\kappa}{U_0^2 + 4\kappa^2 k^2}\cos(kx)\sin(ky) + \frac{\Psi U_0}{U_0^2 + 4\kappa^2 k^2}\sin(kx)\sin(ky), \quad (5.35)$$

and the corresponding effective diffusivity is,

$$\boldsymbol{K} = \frac{1}{2}\frac{\Psi^2 k^2 \kappa}{U_0^2 + 4k^2\kappa^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (5.36)$$

In the limit $Pe \ll 1$ this result is equivalent to the expression we derived above (under the assumption $U_0^2 \ll 4k^2\kappa^2$ or equivalently $Pe^2 U_0^2/4k^2\Psi^2 \ll 1$). In the opposite limit $Pe \gg 1$ we find that the diffusivity asymptotes a constant value smaller than one.