Turbulent motions in the Atmosphere and Oceans

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Course description
The course will present the phenomena, theory, and modeling of turbulence in the Earth's oceans and atmosphere. The scope will range from centimeter to planetary scale motions. The regimes of turbulence will include homogeneous isotropic three dimensional turbulence, convection, boundary layer turbulence, internal waves, two dimensional turbulence, quasi-geostrophic turbulence, and planetary scale motions in the ocean and atmosphere. Prerequisites: the mathematics and physics required for admission to the graduate curriculum in the EAPS department, or consent of the instructor.

Course requirements
Class attendance and discussion, weekly homework assignments.

Reference texts
Andrews, Holton, and Leovy, "Middle atmosphere dynamics"
Frisch, "Turbulence: the legacy of Kolmogorov"
Lesieur, "Turbulence in Fluids", 3rd revised edition
McComb, "The physics of turbulence"
Saffman, "Vortex dynamics"
Salmon, "Lectures on geophysical fluid dynamics"
Tennekes and Lumley, "A first course in Turbulence"
Whitham, "Linear and nonlinear waves"

Chapter 6

Wave mean flow interactions

In the last few weeks we spent quite some time 1) discussing some basic ideas on the transport of tracers by turbulent flows, and 2) investigating the properties of turbulent flows in rotating stratified environments, like the ocean and the atmosphere. The goal of this chapter is to bring together these two bodies of literature to study the interaction of eddy motions with a large scale mean flow in geophysically relevant problems. Following Glenn's lecture on mean field approximations, we will consider eddies generated through instabilities of a zonal mean jet in the quasi-geostrophic approximation. This is a very special example, but it is a useful testbed to develop our intuition about these problems. Furthermore there is no general theory for non-zonal, non-quasi-geostrophic flows.

The literature on eddy mean-slows interactions is so vast that it would be impossible to give a comprehensive review in one lecture. Thus we will select a few topics of particular relevance in the oceanic context. The final goal is to apply these theories to derive closure schemes that represent the effect of eddy motions on mean oceanic and atmospheric flows. The interested reader can find more information in the references given at the end of the chapter.

6.1 The quasi-geostrophic equations on a β-plane

Consider a flow in a Boussinesq fluid with characteristic horizontal length scale L, velocity U, time scale $T \geq L/U$, on a β-plane for which the Coriolis parameter is $f = f_0 + \beta y$. We make the assumption that,

1. the Rossby number $Ro = U/f_0 L$ is small,
2. the β-effect is small, $\beta L/f_0 \leq Ro$,

3. the isopycnal slopes $|\partial_x b|/|\partial_z b|$ and $|\partial_y b|/|\partial_z b|$ are $\leq Ro$ (otherwise vertical motions would not be small),

4. the static stability $N^2 = \partial b/\partial z$ is a function of z only.

Under these assumptions, the leading order equations in Ro give geostrophic balance. Thus we can write the leading order geostrophic velocities in the Ro expansion, as,

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}, \qquad w = 0, \tag{6.1}$$

where ψ is the geostrophic streamfunction,

$$\psi = \frac{p - p_0(z)}{\rho_0 f_0}. \tag{6.2}$$

Hydrostatic balance gives us,

$$\frac{\partial \psi}{\partial z} = \frac{b}{f_0}. \tag{6.3}$$

At the next order in Ro, we obtain the prognostic quasi-geostrophic equations,

$$D_g u - \beta y v - f_0 v_a = \mathcal{G}_x, \tag{6.4}$$
$$D_g v + \beta y u + f_0 u_a = \mathcal{G}_y, \tag{6.5}$$
$$\partial_x u_a + \partial_y v_a + \partial_z w_a = 0, \tag{6.6}$$
$$D_g b + N^2 w_a = \mathcal{B}, \tag{6.7}$$
$$\tag{6.8}$$

where D_g is the time derivative following the geostrophic motions,

$$D_g = \partial_t + u\partial_x + v\partial_y, \tag{6.9}$$

(u_a, v_a, w_a) is the ageostrophic velocity, *i.e* the difference between the actual velocity and the geostrophic one, $(\mathcal{G}_x, \mathcal{G}_y)$ is the external forcing on momentum (*e.g.* wind stress, friction, ...), and $\mathcal{B}$ are the nonconservative buoyancy forces (*e.g.* small scale mixing, sea-surface heat fluxes, ...).

Using (6.4) through (6.7), we can derive the equation for the quasi-geostrophic potential vorticity (QGPV), q,

$$D_g q = \chi, \tag{6.10}$$

where,

$$q = f_0 + \beta y + \partial_x v - \partial_y u + f_0 \partial_z (b/N^2), \tag{6.11}$$
$$\chi = \partial_x \mathcal{G}_y - \partial_y \mathcal{G}_x + f_0 \partial_z (\mathcal{B}/N^2). \tag{6.12}$$

Eq. (6.10) tells us that for conservative flows ($\boldsymbol{\mathcal{G}} = 0$, $\mathcal{B} = 0$) q is conserved following the geostrophic flow. When the flow is not conservative, χ represents the local sources and sinks of q, arising from viscous and diabatic effects. As you can see, the QGPV satisfy the advection-diffusion equation of a generic tracer. Thus we might be able to use the results on tracer transport in turbulent flows to study the dynamics of q.

6.2 Potential vorticity fluxes and the Eliassen-Palm Theorem

The next three sections, up to the definition of Transformed Eulerian Mean, follow very closely the notes of Alan Plumb on eddy-mean flows interactions. If you are interested in learning more on this topic, I encourage you to contact Alan and ask for a copy of his notes.

Consider the small amplitude motions on a steady, zonally-uniform basic state,

$$\bar{u} = \bar{u}(y,t), \qquad \bar{b} = \bar{b}(y,t), \qquad \bar{\psi} = \bar{\psi}(y,t), \tag{6.13}$$

where

$$\bar{u} = -\partial_y \bar{\psi}, \qquad \partial_y \bar{b} = -f_0 \partial_z \bar{u}. \tag{6.14}$$

The mean PV is,

$$\bar{q} = f_0 + \beta y + \partial_y^2 \bar{\psi} + \partial_z \left(\frac{f_0^2}{N^2} \partial_z \bar{\psi} \right). \tag{6.15}$$

The perturbation streamfunction and PV are given by,

$$\psi' = \psi - \bar{\psi}, \qquad q' = q - \bar{q} = \partial_x^2 \psi' + \partial_y^2 \psi' + \partial_z \left(\frac{f_0^2}{N^2} \partial_z \psi' \right) \tag{6.16}$$

Using $v' = \partial_x \psi'$, we can also show that,

$$\overline{v'q'} = \nabla \cdot \boldsymbol{F} = \nabla \cdot \begin{pmatrix} F_y \\ F_z \end{pmatrix} = \nabla \cdot \begin{pmatrix} -\overline{u'v'} \\ \frac{f_0}{N^2} \overline{v'b'} \end{pmatrix}. \tag{6.17}$$

$\boldsymbol{F}$ is known as the Eliassen-Palm flux. Note that the northward component of $\boldsymbol{F}$ is minus the northward flux of zonal momentum by the eddies, $\overline{u'v'}$, while the vertical component is proportional to the northward flux of buoyancy, $\overline{v'b'}$.

Linearizing the quasi-geostrophic potential vorticity equation (6.10), we get,

$$\partial_t q' + \bar{u} \partial_x q' + v' \partial_y \bar{q} = \chi'. \tag{6.18}$$

If we multiply by q' and average, we obtain the eddy potential enstrophy equation,

$$\partial_t \left(\frac{\overline{q'^2}}{2} \right) + \overline{v'q'} \partial_y \bar{q} = \overline{q'\chi'}. \tag{6.19}$$

This equation is the basic ingredient for the Eliassen-Palm theorem: *For waves which are steady ($\partial_t \overline{q'^2} = 0$), of small amplitude, and conservative ($\overline{v'\chi'} = 0$), the northward eddy PV flux vanishes ($\overline{v'q'} = 0$) and the flux $\boldsymbol{F}$ is nondivergent.*

We can now consider the problem of how eddies impact the zonal mean circulation. The mean quasi-geostrophic PV budget reads,

$$\partial_t \bar{q} + \partial_y (\overline{v'q'}) = \bar{\chi}. \tag{6.20}$$

Because of the quasi-geostrophic approximation, eq. (6.20) contains no mean advection term and no vertical component of eddy fluxes.

The influence of the eddies on the mean QGPV, therefore, is entirely described by the northward flux $\overline{v'q'}$. Now we know from the Eliassen-Palm theorem that if the waves are 1) steady, 2) conservative, and 3) of small amplitude, then $\boldsymbol{F}$ is nondivergent and $\overline{v'q'} = 0$. Under these conditions, therefore, the equation for the zonally-averaged QGPV is independent of the eddies. An therefore the full evolution of the mean flow is independent of the eddies. This is known as the *non-acceleration theorem.*

6.3 Mean momentum and buoyancy budgets: conventional approach

In order to fully appreciate the meaning of the Eliassen-Palm theorem, it is useful to consider the zonal mean of the quasi-geostrophic momentum and buoyancy equations,

$$\partial_t \bar{u} - f_0 \bar{v}_a = \bar{\mathcal{G}}_x - \partial_y (\overline{u'v'}), \tag{6.21}$$

$$f_0 \partial_z \bar{u} = -\partial_y \bar{b}, \tag{6.22}$$

$$\partial_y \bar{v}_a + \partial_z \bar{w}_a = 0, \tag{6.23}$$

$$\partial_t \bar{b} + \bar{w}_a N^2 = \bar{\mathcal{B}} - \partial_y (\overline{v'b'}). \tag{6.24}$$

The evolution of the zonal mean state in the presence of eddies is therefore manifested in two terms – the convergence of the eddy flux of momentum, $\overline{u'v'}$, and buoyancy, $\overline{v'b'}$. Both these terms force the mean flow equations and it is important to note that the whole system is coupled, *i.e.*, the buoyancy fluxes can impact on the mean flow, just as much as the momentum fluxes. Thermal wind balance (6.22) links the two. Consider, for example, a wave with $\overline{v'b'} \neq 0$, but $\overline{u'v'} = 0$ (as it is largely true in

the ocean). The mean state cannot respond with a changing mean buoyancy only; thermal wind balance demands a corresponding change in $\bar{u}$. From eq. (6.21), this can only be achieved through an ageostrophic meridional circulation, which would impact on both the momentum and buoyancy budgets. Thus, the eddies will not only drive $\partial_t \bar{u}$ and $\partial_t \bar{b}$, but also $\bar{v}_a$ and $\bar{w}_a$ (except in the unlikely case where the eddy forcing terms conspire not to disturb the thermal wind balance).

Note that the central role of the potential vorticity flux, obvious in the QGPV budget, is not at all obvious here. Indeed, we have seen from the potential enstrophy budget, that, under non-acceleration conditions, $\partial_t \bar{u}$ and $\partial_t \bar{b}$ must be zero. What must, and thus happen, under such circumstances, is that eddies induce an ageostrophic mean motion, which exactly balance the eddy flux terms in (6.21) and (6.24), *i.e.* eddy fluxes induce a mean circulation. This is reminiscent to the result that eddy fluxes of quasi-conserved tracers can have an advective component: in this problem the mean advective effect of the eddy fluxes is represented by the ageostrophic circulation.

6.4 The Transformed Eulerian Mean Theory

The difficulty in interpreting the balance of eddy terms and ageostrophic motions can be overcome by what may seem a mathematical trick, but is in fact linked to the decomposition of eddy fluxes in skew (advective) and symmetric (diffusive) components. The trick is to redefine the mean meridional, ageostrophic, circulation.

Consider the mean buoyancy budget (6.24). This is (apart for the loss of some terms through the quasi-geostrophic assumption) the same as the Eulerian mean budget of a tracer equation. We saw that the eddy flux term can include an advective component. Under quasi-geostrophic assumptions, we can guess what that component is.

We begin by noting that, from eq. (6.23), we may define an ageostrophic mean streamfunction χ_a, such that,

$$(\bar{v}_a, \bar{w}_a) = (-\partial_z \chi_a, \partial_y \chi_a). \tag{6.25}$$

We can then rewrite the mean buoyancy budget in (6.24) as,

$$\partial_t \bar{b} + \partial_y \left(\chi_a + \frac{\overline{v'b'}}{N^2} \right) N^2 = \bar{\mathcal{B}}. \tag{6.26}$$

where we used the fact that $N^2 = N^2(z)$, *i.e.* the vertical stratification does not change with latitude. In this form, it is quite clear that the eddy flux term can be represented as a mean advection, by defining an eddy induced mean streamfunction χ_c as,

$$\chi_c = \frac{\overline{v'b'}}{N^2}. \tag{6.27}$$

We now define the "residual circulation" as,

$$(\bar{v}^\dagger, \bar{w}^\dagger) = (-\partial_z \chi^\dagger, \partial_y \chi^\dagger), \tag{6.28}$$

where the new streamfunction is,

$$\chi^\dagger = \chi_a + \chi_c. \tag{6.29}$$

The streamfunction $\chi^\dagger$ is the so-called *residual streamfunction* and it represents the new definition of mean circulation. It is called a residual circulation, because in many situations χ_a and χ_c tend to oppose each other, and $\chi^\dagger$ is the residual between two strong circulations. If we substitute the definition in (6.27) into the mean buoyancy budget, we obtain,

$$\partial_t \bar{b} + w^\dagger N^2 = \bar{\mathcal{B}}. \tag{6.30}$$

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We thus succeeded in deriving a mean buoyancy equation in which there is no explicit eddy term; buoyancy is transported solely through the mean vertical residual motion. It might be thought, of course, that the eddy terms are still there, implicit in $w^\dagger$. But this was also true of w_a which, as noted earlier, is in general influenced by the eddies. What we have done, is to redefine this influence, so as to put the mean buoyancy budget into its simplest possible form.

We can complete the transformed system of equations,

$$\partial_t \bar{u} - f_0 \bar{v}^\dagger = \bar{\mathcal{G}}_x + \nabla \cdot \boldsymbol{F}, \tag{6.31}$$

$$f_0 \partial_z \bar{u} = -\partial_y \bar{b}, \tag{6.32}$$

$$\partial_y \bar{v}^\dagger + \partial_z \bar{w}^\dagger = 0, \tag{6.33}$$

$$\partial_t \bar{b} + \bar{w}^\dagger N^2 = \bar{\mathcal{B}}, \tag{6.34}$$

where $\boldsymbol{F}$ is the Eliassen-Palm flux.

This transformation makes the role of eddies look quite different–even though the physics described by equations (6.31) through (6.34) is the same described by (6.21) through (6.24). The main advantage is that in terms of $\bar{v}^\dagger$, $\bar{w}^\dagger$, $\partial_t \bar{u}$, and $\partial_t \bar{b}$, the only term representing the eddy forcing is $\nabla \cdot \boldsymbol{F} = \overline{v'q'}$. This eddy forcing appears as an effective body force in the mean momentum equation. It is clear therefore that, under non-acceleration conditions (when $\nabla \cdot \boldsymbol{F} = 0$ and the boundary conditions are independent of eddy dependent terms), $\bar{v}^\dagger$, $\bar{w}^\dagger$, $\partial_t \bar{u}$, and $\partial_t \bar{b}$, are independent of the eddies.

When non-acceleration conditions are not satisfied, the transformed equations offer a more transparent approach to the eddy-mean flow interaction problem, simply because the single term represented by the effective force $\nabla \cdot \boldsymbol{F}$ entirely describes the

eddy forcing of the mean state. In fact, this formulation gives us another interpretation of $\boldsymbol{F}$, as an eddy flux of transformed negative (easterly) momentum, which is a more reliable measure of eddy transport of momentum than $\overline{u'v'}$ itself.

The interpretation of $\boldsymbol{F}$ as a momentum flux may seem to be a result of mathematical tinkering. However, it should be remembered that the process of taking a mean is an arbitrary one–there is no unique way of doing it. Thus, it is legitimate to choose the definition of mean that simplifies the most the problem at hand. The Transformed Eulerian Mean equations indeed give us a clearer picture on what is going on when eddies interact with a mean flow.

6.5 Wave mean flow interactions in the Eady problem

This section is part of notes I have written in collaboration with Geoff Vallis. These notes are now part of Geoff's book on atmospheric and ocean fluid dynamics.

Eady proposed a very simple configuration to study the evolution of baroclinic instability in a channel. I assume that you are all familiar with the basic formulation of the problem. Eady made considered the instability of a meridional buoyancy front in a re-entrant channel under the following assumptions: (i) The instability is described by the quasi-geostrophic equations. (ii) The motion is on the f-plane. (iii) The fluid is uniformly stratified. That is, N^2 is a constant. (iv) The basic state is of uniform shear. That is $u_0 = Uz/D$ where U is a constant and z the height co-ordinate and D the domain depth. (v) The motion is contained between two rigid, flat horizontal surfaces.

We now consider the eddy fluxes in the Eady problem and how these might feed back onto the mean flow. Because of the simplicity of the setting the problem can be fully solved in both the Eulerian or residual frameworks, and it is therefore an instructive example of the usefulness of the TEM methodology.

6.5.1 Formulation

Let us first distinguish between the basic flow, the zonal mean fields, and the perturbation. The basic flow is the flow around which the equations of motion are linearized; this flow is unstable, and the perturbations, assumed small, grow exponentially with time. Because they are (formally) always small they do not affect the basic flow, but they do produce changes in the zonal mean velocity and buoyancy fields. In Eulerian

form this is represented by,

$$\partial_t \bar{u} - f_0 \bar{v}_a = -\partial_y \overline{u'v'}, \qquad \bar{b}_t + N^2 \bar{w}_a = -\partial_y \overline{v'b'}. \tag{6.35}$$

The TEM version of these equations as we have shown in the previous section, is,

$$\partial_t \bar{u} - f_0 \bar{v}^\dagger = \overline{v'q'}, \qquad \bar{b}_t + N^2 \bar{w}^\dagger = 0, \tag{6.36}$$

where in the Eady problem $\partial_y \overline{u'v'}$ and $\overline{v'q'}$ are both zero. This follows without detailed calculation, by noting that the eddy potential vorticity flux is zero because the basic state has zero potential vorticity and therefore none may be generated. Further, because the basic state does not vary in y the there can be no momentum flux convergence in the y-direction, and so the momentum flux itself is zero if it is zero on the boundary. We can calculate the perturbation quantities from the solution to Eady problem (e.g., calculate $\overline{u'v'}$) and thus infer the structure of the mean flow tendencies $\bar{u}_t$ and $\bar{b}_t$ and the meridional circulation, $(\bar{v}, \bar{w})$ or $(\bar{v}^\dagger, \bar{w}^\dagger)$. All of these fields are perturbation quantities and all are exponentially growing, and so in reality they will eventually have a finite effect on the pre-existing zonal flow, but in the Eady problem, or any similar linear problem, such rectification is assumed small and neglected.

Using the thermal wind relation, $f_0 \bar{u}_z = -\bar{b}_y$ to eliminate time derivatives in (6.35) gives an equation for the meriodional streamfunction χ_a, namely,

$$f_0^2 \partial_{zz} \chi_a + N^2 \partial_{yy} \chi_a = -\partial_{yy} \overline{v'b'}, \tag{6.37}$$

The boundary conditions are that $\chi_a = 0$ at $y = 0, L$ and $z = 0, D$, where L is the width of the channel and D the depth. Similarly we obtain an equation for the residual streamfunction,

$$f_0^2 \partial_{zz} \chi^\dagger + N^2 \partial_{yy} \chi^\dagger = 0, \tag{6.38}$$

where now the boundary conditions are that $N^2 \bar{w}^\dagger = \partial_y \overline{v'b'}$ at the upper and lower boundaries, and $v^\dagger = 0$ at the lateral boundaries. In terms of the residual streamfunction this is,

$$\chi^\dagger = \frac{\overline{v'b'}}{N^2}, \text{ at } z = 0, D, \qquad \chi^\dagger = 0, \text{ at } y = 0, L. \tag{6.39}$$

The residual and overturning circulations are related by (6.29), and (6.37) and (6.38) are, at one level, simply different representations of the same problem, connected by a simple mathematical transformation. However, the residual streamfunction better represents the total transport of the fluid. Eq. (6.38) is particularly simple, because of the absence of potential vorticity fluxes in the interior, and it is apparent that the residual circulation is driven by boundary sources. We care only about the spatial structure of the right-hand sides of (6.38) and the boundary conditions of (6.39). The former is given by,

$$-\partial_{yy} \overline{v'b'} \propto -\partial_{yy} \sin^2 ly = -2l^2 \cos 2ly. \tag{6.40}$$

The eddy heat fluxes in the Eady problem are therefore independent of height. We have already mentioned that the momentum fluxes are identically zero. Hence the EP fluxes $\boldsymbol{F} = (-\overline{u'v'}, f_0/N^2\overline{v'b'})$ is directed purely vertically. And the boundary conditions for the residual circulation are,

$$\chi^\dagger(y,0) = \chi^\dagger(y,1) \propto \sin^2 ly. \tag{6.41}$$

6.5.2 Solution

The solutions to (6.37) and (6.38) may be obtained either analytically or numerically. The Eulerian circulation is dominated by a single cell, with equatorwards motion aloft and polewards motion near the surface. This suggests that heat flux convergence in high latitudes is leading to mean rising motion, with the precise shape of the streamfunction determined by the need to satisfy the boundary conditions. Although this is true, the heat flux arises because of the motion of fluid parcels, so it may be a little misleading to infer, as one might from the Eulerian streamfunction, that the heat flux causes the individual parcels to rise or sink in this fashion. The residual streamfunction is a better indicator of the total mass transport and, perhaps as one might intuitively expect, these show parcels rising in the low latitudes and sinking in high latitudes, providing a tendency to flatten the isopycnals and reduce the meridional temperature gradient.

The residual circulation also shows fluid entering of leaving the domain at the boundary what does this represent? Suppose that instead of solving the continuous problem we had posed the problem in a finite number of layers . As the number of layers increases the solutions to the linear baroclinic instability problem approach that of the Eady problem; however, the residual circulation is closed in the layered model, and the sum over all the layers of the meridional transport vanishes. Now, in the layered model the vertical boundary conditions are built in to the representation by way of a redefinition of potential vorticity of the top and bottom layer, so that, in the layered version of Eady problem there appears to be a potential vorticity gradient in these two layers, instead of a buoyancy gradient at the boundary. The residual circulation is then closed by a return flow that occurs only in the top and bottom layers, and as the number of layers increases this flow is confined to a thinner and thinner layer, and to a deltafunction in the continuous limit.

The effect on the mean flow is inferred directly from the residual circulation: the mean flow acceleration is proportional to $v^\dagger$ and the buoyancy tendency is proportional to $w^\dagger$. Because there is no momentum flux convergence in the problem the zonal flow tendency is entirely baroclinic its vertical integral is zero and over most of the domain is such as to reduce the mean shear. Consistently (using thermal wind) the buoyancy tendency is such as to reduce the meridional temperature gradient; that is, the instabilities act to transport heat polewards and reduce the instability of the

mean flow.

6.6 Parameterizing mesoscale motions in numerical models

So far we avoided getting our hands dirty to find closures that relate the eddy fluxes to the mean flow. The TEM formalism is however often invoked to derive parameterizations of the interaction between large-scale mean flows and small-scale transient eddy motions. In this section, we will use the results of TEM together with some physical insight to derive sets of equations where the eddy terms do not appear explicitly. Two approaches are used in the literature, one based on an energetic argument, and the other on potential vorticity mixing theory.

6.6.1 The energetic argument

The energetic argument has been used to derive eddy parameterizations in the ocean only. Thus we will restrict our scope to ocean dynamics in this section.

Although mesoscale eddy motions can be directly generated by external forcing, like the wind field, most of the mesoscale eddy energy is believed to be the result of instabilities in many forms (Pedlosky, 1987). The common belief is that eddies are generated by extracting kinetic and potential energy from the mean flow. This might not be the whole story though: in two dimensional and quasi-geostrophic turbulence, eddy motions can create an inverse energy cascade and return some of the energy back to the mean flow. The point here is that an analysis of the exchange of energy between mean and eddy motions might be fruitful to derive parameterizations.

The total mechanical energy is given by the sum of the kinetic K and potential energies P, which in the geostrophic approximation are,

$$K = \frac{1}{2}(u^2 + v^2), \qquad P = \frac{1}{2}\frac{b^2}{N^2}. \tag{6.42}$$

Conservation of total energy takes the form,

$$\left[\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right][K + P] + \frac{1}{\rho_0}\nabla \cdot (p\boldsymbol{u}_a) = \boldsymbol{u} \cdot \boldsymbol{\mathcal{G}} + \frac{b\mathcal{B}}{N^2}. \tag{6.43}$$

Exchange of energy between eddies and a zonal flow may be defined following the separation of the zonally averaged kinetic and potential energies into components

associated with the eddy and mean motions. In the quasi-geostrophic approximation, this is straightforward,

$$K_M = \tfrac{1}{2}(\bar{u}^2 + \bar{v}^2), \qquad P_M = \frac{1}{2}\frac{\bar{b}^2}{N^2}, \tag{6.44}$$

$$K_E = \tfrac{1}{2}\left(\overline{u'^2} + \overline{v'^2}\right), \qquad P_E = \frac{1}{2}\frac{\overline{b'^2}}{N^2}. \tag{6.45}$$

Let us assume once again that the basic state is a zonal flow, *i.e.* $\bar{u} = \bar{u}(y, z, t)$, $\bar{b} = \bar{b}(y, z, t)$, but $\bar{v} = \bar{w} = 0$. The equations for the mean kinetic and potential energies are,

$$\frac{\partial K_M}{\partial t} + \frac{1}{\rho_0}\nabla \cdot (\bar{\boldsymbol{u}}_a \bar{p}) = \bar{b}\bar{w}_a - \bar{u}\partial_y(\overline{u'v'}) + \bar{u}\bar{\mathcal{G}}, \tag{6.46}$$

$$\frac{\partial P_M}{\partial t} + \bar{b}\bar{w}_a = -\bar{b}\partial_y\left(\frac{\overline{v'b'}}{N^2}\right) + \frac{\bar{b}\bar{\mathcal{B}}}{N^2}. \tag{6.47}$$

The eddy terms on the rhs represent conversion of mean energy into turbulent energy and are often associated with instabilities of the mean flow.

Equations (6.46) and (6.47) can be combined together in the form,

$$\begin{aligned}\frac{\partial}{\partial t}(K_M + P_M) + \frac{1}{\rho_0}\nabla \cdot (\bar{\boldsymbol{u}}_a \bar{p}) = \\ = -\partial_y\left(\bar{u}\overline{u'v'} + \bar{b}\frac{\overline{v'b'}}{N^2}\right) + \overline{u'v'}\partial_y\bar{u} + \frac{\overline{v'b'}}{N^2}\partial_y\bar{b} + \bar{u}\bar{\mathcal{G}} + \frac{\bar{b}\bar{\mathcal{B}}}{N^2}.\end{aligned} \tag{6.48}$$

The ocean is a strongly stratified fluid and most of the energy in the basic state is stored as potential energy due to tilted isopycnal surfaces. This energy is converted into mesoscale eddy motions mainly through baroclinic instabilities. Thus in equation (6.48) the buoyancy eddy flux terms typically dominate over the eddy momentum flux terms.

The divergent terms represent transport of eddy activity from one region to another. In a statistically steady state, we can assume that there is no net transport of mean mechanical energy out of the domain considered. Neglecting the kinetic energy loss terms and the divergent terms, we have that on average,

$$\frac{\partial}{\partial t}(K_M + P_M) \sim \frac{\overline{v'b'}}{N^2}\partial_y\bar{b} + \text{external forcing}. \tag{6.49}$$

Baroclinic instability tends to extract potential energy from the mean state. The simplest closure that would ensure that energy is always released from the mean state and lost into the eddy filed is,

$$\overline{v'b'} = -K\partial_y\bar{b}. \tag{6.50}$$

This closure scheme was first proposed by Gent and McWilliams in 1990, and it is now in use in most coarse-resolution ocean models.

In terms of the TEM, this closure provide an estimate of the eddy induced circulation,

$$\chi_c = -K \frac{\partial_y \bar{b}}{N^2} \tag{6.51}$$

The parameterization of Gent and McWilliams is thus equivalent to assuming that the eddy induced circulation is proportional to the isopycnal slope. As long as mean isopycnals are tilted, there is available potential energy to drive an eddy-induced circulation.

In terms of the Transformed eulerian mean formalism the parameterization of Gent and McWilliams is as a closure for the eddy forcing of the residual circulation, *i.e.*,

$$\nabla \cdot \boldsymbol{F} \approx -f_0 \partial_z \left[K \frac{\partial_y \bar{b}}{N^2} \right]. \tag{6.52}$$

In this closure the eddy stress is proportional to the isopycnal slope. In order to satisfy conservation of mean momentum, it is customary to impose $K = 0$ at the ocean surface and ocean bottom.

Further reading

Bryan K, Dukowicz JK, Smith RD, 1999: On the mixing coefficient in the parameterization of bolus velocity *J. Phys. Oceanogr.***29**, 2442-2456.

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6.6.2 The potential vorticity mixing argument

Numerous studies suggest that mesoscale eddies tend to mix potential vorticity along isopycnals (Holland and Rhines 1980, Rhines 1982, Marshall 2000). In the quasi-geostrophic approximation isopycnals are to leading order flat, and mixing occurs along horizontal surfaces. Thus it might appear that an appropriate closure hypothesis is to assume that PV is fluxed down its mean gradient. Unfortunately direct numerical simulations of quasi-geostrophic turbulence show that PV eddy fluxes are directed along mean PV contours, not across them. The conundrum turns out to be related to the difference between the skew and the diffusive flux, and the difference between PV and QGPV.

Let us consider the eddy buoyancy flux in the quasi-geostrophic approximation. We wish to relate the Transformed Eulerian Mean approach to the theory of tracer transport. This will allow a better understanding of the role of buoyancy and PV fluxes in the quasi-gesotrophic approximation. First decompose the flux into skew and symmetric components as,

$$\begin{pmatrix} \overline{v'b'} \\ \overline{w'b'} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\overline{v'b'}}{N^2} \\ -\frac{\overline{v'b'}}{N^2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ N^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\overline{w'b'}N^2 + \overline{v'b'}\overline{b}_y}{N^2} \end{pmatrix} \tag{6.53}$$

$$= \begin{pmatrix} 0 & \chi_c \\ -\chi_c & 0 \end{pmatrix} \begin{pmatrix} 0 \\ N^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{\overline{w'b'}N^2 + \overline{v'b'}\overline{b}_y}{N^2}. \end{pmatrix} \begin{pmatrix} 0 \\ N^2 \end{pmatrix} \tag{6.54}$$

The first bracket the Stokes drift generated by eddy transport along isopycnals; in the quasi-geostrophic approximation the isopycnals are flat to leading order, therefore the skew flux is due to the horizontal eddy flux. The second bracket represents the symmetric flux: in the quasi-geostrophic approximation this flux is vertical and is directed across the vertical gradient N^2. It can be shown that the symmetric flux is likely to be diffusive. Consider the buoyancy variance equation,

$$\partial_t \overline{b'^2} + \nabla \cdot \overline{\boldsymbol{u}'b'^2} = -2\overline{v'b'}\overline{b}_y - 2\overline{w'b'}N^2 + 2D, \tag{6.55}$$

where D represents all diabatic terms. In steady state and for homogenous eddy statistics we have a balance between the generation of eddy variance by eddies and the destruction by dissipation. This implies that the symmetric flux must be down the vertical gradient, i.e. diffusive. However this flux is not of interest in quasi-geostropic theories because it is $O(Ro^2)$. What matters is that by going into the Transformed Eulerian Mean framework we include the Stokes drift into the definition of mean velocity, and thus we eliminate the skew flux from the buoyancy equation. As we have seen, this trick eliminates all eddy flux terms from the buoyancy equation. This is true only in the quasi-geostrophic approximation, because the symmetric fluxes are of higher order. In general by going into the Transformed Eulerian Mean framework, one eliminates only the skew component of the eddy flux from the buoyancy budget, but not the symmetric component.

With a better understanding of the buoyancy flux decomposition, we can now proceed to analyze the PV budget. The full Ertel potential vorticity in the quasi-geostrophic approximation is given by $P = fN^2 + fb_z + \zeta N^2$. The eddy PV flux can be computed easily,

$$\begin{aligned} \overline{\boldsymbol{u}'P'} &= f\overline{\boldsymbol{u}'b'_z} + \overline{\boldsymbol{u}'\zeta'}, && (6.56)\\ &= \left[f\overline{v'b'_z} + \overline{v'\zeta'}\right]\boldsymbol{j} + O(Ro^2). && (6.57) \end{aligned}$$

Thus to leading order the eddy PV flux has only an horizontal component. It is instructive to write the meridonal flux in the form,

$$\begin{aligned} \overline{v'P'} &= f\partial_z(\overline{v'b'}) + \overline{v'\zeta'} && (6.58)\\ &= f\frac{\partial}{\partial z}\left(\frac{\overline{v'b'}}{N^2}N^2\right) + \overline{v'\zeta'} && (6.59)\\ &= f\frac{\overline{v'b'}}{N^2}\partial_z(N^2) + fN^2\frac{\partial}{\partial z}\left(\frac{\overline{v'b'}}{N^2}\right). && (6.60) \end{aligned}$$

Using the fact that $\bar{P}_z = \partial_z(fN^2) + O(Ro)$ and that $v'_z = f^{-1}b'_x + O(Ro)$ we can simplify the expression as,

$$\overline{v'P'} = \frac{\overline{v'b'}}{N^2}\bar{P}_z + N^2\overline{v'q'} + O(Ro^2). \quad (6.61)$$

Using this relation we can write the full Ertel PV flux as the sum of skew and symmetric components,

$$\begin{pmatrix} \overline{v'P'} \\ \overline{w'P'} \end{pmatrix} = \begin{pmatrix} 0 & \chi_c \\ -\chi_c & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \bar{P}_z \end{pmatrix} + \begin{pmatrix} \overline{v'P'} - \frac{\overline{v'b'}}{N^2}\bar{P}_z \\ \overline{w'P'} + \frac{\overline{v'b'}}{N^2}\bar{P}_y \end{pmatrix}. \quad (6.62)$$

To leading order this relation is,

$$\begin{pmatrix} \overline{v'P'} \\ \overline{w'P'} \end{pmatrix} = \begin{pmatrix} 0 & \chi_c \\ -\chi_c & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \bar{P}_z \end{pmatrix} + N^2\begin{pmatrix} \overline{v'q'} \\ 0 \end{pmatrix} \quad (6.63)$$

In this form we see that the Ertel PV flux is composed of an advective skew component and a residual component, which happens to be proportional to the QGPV. In the lecture on passive tracer transport we emphasized that the residual flux tends to mix across tracer contours, while the skew component does not mix. Furthermore the skew component is typically much larger than the residual component. Similarly in this problem, the skew component advects PV around and dominates the full PV flux, but it is the residual flux that achieves mixing. This can be demonstrated by considering the QGPV variance budget,

$$\partial_t \overline{q'^2} + \partial_y \overline{v'q'^2} = -2\overline{v'q'}\bar{q}_y + 2D. \tag{6.64}$$

this suggests that in steady state for homogeneous turbulence, the eddy QGPV flux must be down its mean gradient,

$$\overline{v'q'} = -K\bar{q}_y. \tag{6.65}$$

Plugging this closure in the expression for the Eliassen-Palm fluxes gives,

$$\nabla \cdot \boldsymbol{F} = \overline{v'q'} = -K\left[\beta - \partial_{yy}\bar{u} + f_0\partial_z\left(\frac{\partial_y \bar{b}}{N^2}\right)\right]. \tag{6.66}$$

This expression for the eddy forcing of the residual circulation differs from that in (6.52). The two expressions are equivalent if 1) K is constant, 2) there is no planetary PV gradient ($\beta = 0$), and 3) PV is dominated by the baroclinic term. In the ocean condition 3 is often satisfied. Condition 2 is harder to assess, because it depends on whether eddies mix on distances large enough to feel the effect of β. Condition 1 instead cannot be satisfied, because one needs to impose $K = 0$ at the boundaries in the Gent-McWilliams parameterization and therefore K cannot be constant. Thus the two closure schemes are different. It is open to debate which approach is more appropriate. A good discussion can be found in the paper by Treguier et al. (1997).

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