12.820 - Turbulence in the Atmosphere and Oceans

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Macrtoturbulence Mean Flow Interactions

Glenn discussed under what approximations we can relate the tracer fluxes generated by a turbulent flow to the large scale tracer gradient. In this lecture we show how to apply those ideas to study the effect of macroturbulence on the a large scale circulation of the ocean and atmosphere. As a first step we will consider eddies generated through instabilities of a zonal barotropic mean jet. Next we will consider eddies generated through barolcinic instabilities of a zonal jet in the quasi-geostrophic approximation. These are very special example, but they are useful and simple testbed to develop our intuition about these problems.

Eddy mean flow interactions in a barotropic jet

Glenn's discussion of eddy diffusivity is valid only for tracers that satisfy an advection diffusion equation. Is it possible to extend the argument to momentum and define an *eddy viscosity*? This problem is discussed at length in the notes of Alan Plumb on "Eddy transport in the atmosphere and the ocean", and we follow closely that presentation.

Let us consider, for simplicity, a barotropic velocity field (that is a 2D system). The absolute vorticity $\zeta_a = f + v_x - u_y$ satisfies an advection-diffusion equation of the form,

$$\partial_t \zeta_a + \boldsymbol{u} \cdot \nabla \zeta_a = F_y - E_x. \tag{1}$$

where E and F represent friction or other forces. Momentum does not satisfy a conservation equation, because of nonlocal pressure terms. As we are about to show, we can use the conservation of absolute vorticity to make some sense of momentum transport.

Let's start with the zonal momentum equation,

$$u_t + uu_x + v\ (u_y - f) = -\frac{1}{\rho_0} p_x + F, \tag{2}$$

where F is the zonal acceleration due to friction or other forces. We now want to consider the zonal average of eq. (2). The zonal mean is defined as,

$$\bar{u}(y,t) \equiv \frac{1}{L} \int_{-L/2}^{L/2} a(x,y,t)\ \mathrm{d}x. \tag{3}$$

Note that,

$$\bar{u}_x = 0, \qquad \bar{v}_y = 0, \qquad \bar{p}_x = 0. \tag{4}$$

The zonal average of eq. (2) is then,

$$\bar{u}_t + \bar{v}\ (\bar{u}_y - f) = -\partial_y \overline{u'v'} + \bar{F}. \tag{5}$$

The eddy term is due to the northward flux of zonal momentum $\overline{u'v'}$. Because of pressure gradients, momentum is not conserved by eddy motions and we will see that there is no basis to expect a relationship of the form,

$$\overline{u'v'} = -D\partial_y \bar{u}. \tag{6}$$

But vorticity satisfies an advection diffusion equation and we can use the results on scalar turbulence derived in the previous section. We showed that for scalars, it is possible to relate the eddy fluxes to the mean tracer distribution. Thus we might expect that there is a relationship between absolute vorticity fluxes and the mean vorticity gradient. This relationship can be further used to relate the momentum fluxes to a mean gradient. Note in fact that,

$$\overline{v'\zeta'} = \overline{v'v'_x} - \overline{v'u'_y} \tag{7}$$

$$= -\partial_y \overline{u'v'}. \tag{8}$$

Thus we can rewrite the zonal mean momentum equation as,

$$\bar{u}_t + \bar{v}\ (\bar{u}_y - f) = \overline{v'\zeta'} + \bar{F}. \tag{9}$$

So that, whereas in eq. (5) the eddies appear as an agency of momentum transport through the eddy momentum flux $\overline{u'v'}$, in eq. (9) they appear as an eastward body force, equal to the northward eddy flux of vorticity, acting on the mean.

The zonal mean vorticity equation reads,

$$\bar{\zeta}_t + \bar{v}\bar{\zeta}_y = \bar{F}_y - \partial_y \overline{v'\zeta'}, \tag{10}$$

and, by appealing to the arguments given in the previous section we can write,

$$\overline{v'\zeta'} = -D\partial_y \bar{\zeta}. \tag{11}$$

This relationship can used to derive an expression for the eddy momentum flux,

$$\partial_y \overline{u'v'} = D\partial_y \bar{\zeta} = D\left(\beta - \bar{u}_{yy}\right). \tag{12}$$

If $\beta = 0$ and D is uniform, then we can integrate this expression (assuming vanishing flux at the boundaries),

$$\overline{u'v'} = -D\ \partial_y \bar{u}. \tag{13}$$

It seems like we managed to recover a diffusive closure for momentum! However this not quite so, because the assumptions we had to make are a little suspect. D cannot be uniform: for example the vorticity flux must vanish at the boundaries where $v' = 0$. Moreover, if β is nonzero, then the whole approach does not work and mean velocity gradients might not be important at all. The main conclusion is that turbulent fluctuations cannot be represented as an enhanced viscosity in the momentum equation. But not all is lost, because progress can still be made by considering the vorticity flux, rather than the momentum flux.

Eddy mean flow interactions in a baroclinic jet

The eddy feedback on baroclinic jets can also be expressed through the flux of a conserved quantity under some approximation. This is the goal of this section.

A primer for the quasi-geostrophic equations on a β-plane

Consider a flow in a Boussinesq fluid with characteristic horizontal length scale L, velocity U, time scale $T \geq L/U$, on a β-plane for which the Coriolis parameter is $f = f_0 + \beta y$. We make the assumption that,

1. the Rossby number $Ro = U/f_0 L$ is small,
2. the β-effect is small, $\beta L/f_0 \leq Ro$,

3. the isopycnal slopes $|\partial_x b|/|\partial_z b|$ and $|\partial_y b|/|\partial_z b|$ are $\leq Ro$ (otherwise vertical motions would not be small),

4. the static stability $N^2 = \partial b/\partial z$ is a function of z only.

Under these assumptions, the leading order equations in Ro give geostrophic balance. Thus we can write the leading order geostrophic velocities in the Ro expansion, as,

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}, \qquad w = 0, \tag{14}$$

where ψ is the geostrophic streamfunction,

$$\psi = \frac{p - p_0(z)}{\rho_0 f_0}. \tag{15}$$

Hydrostatic balance gives us,

$$\frac{\partial \psi}{\partial z} = \frac{b}{f_0}. \tag{16}$$

At the next order in Ro, we obtain the prognostic quasi-geostrophic equations,

$$D_g u - \beta y v - f_0 v_a = \mathcal{G}_x, \tag{17}$$
$$D_g v + \beta y u + f_0 u_a = \mathcal{G}_y, \tag{18}$$
$$\partial_x u_a + \partial_y v_a + \partial_z w_a = 0, \tag{19}$$
$$D_g b + N^2 w_a = \mathcal{B}, \tag{20}$$
$$\tag{21}$$

where D_g is the time derivative following the geostrophic motions,

$$D_g = \partial_t + u\partial_x + v\partial_y, \tag{22}$$

(u_a, v_a, w_a) is the ageostrophic velocity, *i.e* the difference between the actual velocity and the geostrophic one, $(\mathcal{G}_x, \mathcal{G}_y)$ is the external forcing on momentum (*e.g.* wind stress, friction, ...), and $\mathcal{B}$ are the nonconservative buoyancy forces (*e.g.* small scale mixing, sea-surface heat fluxes, ...).

Using (17) through (20), we can derive the equation for the quasi-geostrophic potential vorticity (QGPV), q,

$$D_g q = \chi, \tag{23}$$

where,

$$q = f_0 + \beta y + \partial_x v - \partial_y u + f_0 \partial_z (b/N^2), \tag{24}$$
$$\chi = \partial_x \mathcal{G}_y - \partial_y \mathcal{G}_x + f_0 \partial_z (\mathcal{B}/N^2). \tag{25}$$

Eq. (23) tells us that for conservative flows ($\boldsymbol{\mathcal{G}} = 0$, $\mathcal{B} = 0$) q is conserved following the geostrophic flow. When the flow is not conservative, χ represents the local sources and sinks of q, arising from viscous and diabatic effects. As you can see, the QGPV satisfy the advection-diffusion equation of a generic tracer. Thus we might be able to use the results on tracer transport in turbulent flows to study the dynamics of q.

Potential vorticity fluxes and the Eliassen-Palm Theorem

The next three sections, up to the definition of Transformed Eulerian Mean, follow very closely the notes of Alan Plumb on eddy-mean flows interactions. If you are interested in learning more on this topic, I encourage you to contact Alan and ask for a copy of his notes.

Consider the small amplitude motions on a steady, zonally-uniform basic state,

$$\bar{u} = \bar{u}(y,t), \qquad \bar{b} = \bar{b}(y,t), \qquad \bar{\psi} = \bar{\psi}(y,t), \tag{26}$$

where

$$\bar{u} = -\partial_y \bar{\psi}, \qquad \partial_y \bar{b} = -f_0 \partial_z \bar{u}. \tag{27}$$

The mean PV is,

$$\bar{q} = f_0 + \beta y + \partial_y^2 \bar{\psi} + \partial_z \left(\frac{f_0^2}{N^2} \partial_z \bar{\psi} \right). \tag{28}$$

The perturbation streamfunction and PV are given by,

$$\psi' = \psi - \bar{\psi}, \qquad q' = q - \bar{q} = \partial_x^2 \psi' + \partial_y^2 \psi' + \partial_z \left(\frac{f_0^2}{N^2} \partial_z \psi' \right) \tag{29}$$

Using $v' = \partial_x \psi'$, we can also show that,

$$\overline{v'q'} = \nabla \cdot \boldsymbol{F} = \nabla \cdot \begin{pmatrix} F_y \\ F_z \end{pmatrix} = \nabla \cdot \begin{pmatrix} -\overline{u'v'} \\ \frac{f_0}{N^2} \overline{v'b'} \end{pmatrix}. \tag{30}$$

$\boldsymbol{F}$ is known as the Eliassen-Palm flux. Note that the northward component of $\boldsymbol{F}$ is minus the northward flux of zonal momentum by the eddies, $\overline{u'v'}$, while the vertical component is proportional to the northward flux of buoyancy, $\overline{v'b'}$.

Linearizing the quasi-geostrophic potential vorticity equation (23), we get,

$$\partial_t q' + \bar{u} \partial_x q' + v' \partial_y \bar{q} = \chi'. \tag{31}$$

If we multiply by q' and average, we obtain the eddy potential enstrophy equation,

$$\partial_t \left(\frac{\overline{q'^2}}{2} \right) + \overline{v'q'} \partial_y \bar{q} = \overline{q'\chi'}. \tag{32}$$

This equation is the basic ingredient for the Eliassen-Palm theorem: *For waves which are steady* ($\partial_t \overline{q'^2} = 0$)*, of small amplitude, and conservative* ($\overline{v'\chi'} = 0$)*, the northward eddy PV flux vanishes* ($\overline{v'q'} = 0$) *and the flux* $\boldsymbol{F}$ *is nondivergent.*

We can now consider the problem of how eddies impact the zonal mean circulation. The mean quasi-geostrophic PV budget reads,

$$\partial_t \bar{q} + \partial_y(\overline{v'q'}) = \bar{\chi}. \tag{33}$$

Because of the quasi-geostrophic approximation, eq. (33) contains no mean advection term and no vertical component of eddy fluxes.

The influence of the eddies on the mean QGPV, therefore, is entirely described by the northward flux $\overline{v'q'}$. Now we know from the Eliassen-Palm theorem that if the waves are 1) steady, 2) conservative, and 3) of small amplitude, then $\boldsymbol{F}$ is nondivergent and $\overline{v'q'} = 0$. Under these conditions, therefore, the equation for the zonally-averaged QGPV is independent of the eddies. An therefore the full evolution of the mean flow is independent of the eddies. This is known as the *non-acceleration theorem.*

Mean momentum and buoyancy budgets: conventional approach

In order to fully appreciate the meaning of the Eliassen-Palm theorem, it is useful to consider the zonal mean of the quasi-geostrophic momentum and buoyancy equations,

$$\partial_t \bar{u} - f_0 \bar{v}_a = \bar{\mathcal{G}}_x - \partial_y(\overline{u'v'}), \tag{34}$$

$$f_0 \partial_z \bar{u} = -\partial_y \bar{b}, \tag{35}$$

$$\partial_y \bar{v}_a + \partial_z \bar{w}_a = 0, \tag{36}$$

$$\partial_t \bar{b} + \bar{w}_a N^2 = \bar{\mathcal{B}} - \partial_y(\overline{v'b'}). \tag{37}$$

The evolution of the zonal mean state in the presence of eddies is therefore manifested in two terms – the convergence of the eddy flux of momentum, $\overline{u'v'}$, and buoyancy, $\overline{v'b'}$. Both these terms force the mean flow equations and it is important to note that the whole system is coupled, *i.e.*, the buoyancy fluxes can impact on the mean flow, just as much as the momentum fluxes. Thermal wind balance (35) links the two. Consider, for example, a wave with $\overline{v'b'} \neq 0$, but $\overline{u'v'} = 0$ (as it is largely true in the ocean). The mean state cannot respond with a changing mean buoyancy only; thermal wind balance demands a corresponding change in $\bar{u}$. From eq. (34), this can only be achieved through an ageostrophic meridional circulation, which would impact on both the momentum and buoyancy budgets. Thus, the eddies will not only drive $\partial_t \bar{u}$ and $\partial_t \bar{b}$, but also $\bar{v}_a$ and $\bar{w}_a$ (except in the unlikely case where the eddy forcing terms conspire not to disturb the thermal wind balance).

Note that the central role of the potential vorticity flux, obvious in the QGPV budget, is not at all obvious here. Indeed, we have seen from the potential enstrophy budget, that, under non-acceleration conditions, $\partial_t \bar{u}$ and $\partial_t \bar{b}$ must be zero. What must, and thus happen, under such circumstances, is that eddies induce an ageostrophic mean motion, which exactly balance the eddy flux terms in (34) and (37), *i.e.* eddy fluxes induce a mean circulation. This is reminiscent to the result that eddy fluxes of quasi-conserved tracers can have an advective component: in this problem the mean advective effect of the eddy fluxes is represented by the ageostrophic circulation.

The Transformed Eulerian Mean Theory

The difficulty in interpreting the balance of eddy terms and ageostrophic motions can be overcome by what may seem a mathematical trick, but is in fact linked to the decomposition of eddy fluxes in skew (advective) and symmetric (diffusive) components. The trick is to redefine the mean meridional, ageostrophic, circulation.

Consider the mean buoyancy budget (37). This is (apart for the loss of some terms through the quasi-geostrophic assumption) the same as the Eulerian mean budget of a tracer equation. We saw that the eddy flux term can include an advective component. Under quasi-geostrophic assumptions, we can guess what that component is.

We begin by noting that, from eq. (36), we may define an ageostrophic mean streamfunction χ_a, such that,

$$(\bar{v}_a, \bar{w}_a) = (-\partial_z \chi_a, \partial_y \chi_a). \tag{38}$$

We can then rewrite the mean buoyancy budget in (37) as,

$$\partial_t \bar{b} + \partial_y \left(\chi_a + \frac{\overline{v'b'}}{N^2} \right) N^2 = \bar{\mathcal{B}}. \tag{39}$$

where we used the fact that $N^2 = N^2(z)$, *i.e.* the vertical stratification does not change with latitude. In this form, it is quite clear that the eddy flux term can be represented as a mean advection, by defining an eddy induced mean streamfunction χ_c as,

$$\chi_c = \frac{\overline{v'b'}}{N^2}. \tag{40}$$

We now define the "residual circulation" as,

$$(\bar{v}^\dagger, \bar{w}^\dagger) = (-\partial_z \chi^\dagger, \partial_y \chi^\dagger), \tag{41}$$

where the new streamfunction is,

$$\chi^\dagger = \chi_a + \chi_c. \tag{42}$$

The streamfunction $\chi^\dagger$ is the so-called *residual streamfunction* and it represents the new definition of mean circulation. It is called a residual circulation, because in many situations χ_a and χ_c tend to oppose each other, and $\chi^\dagger$ is the residual between two strong circulations. If we substitute the definition in (40) into the mean buoyancy budget, we obtain,

$$\partial_t \bar{b} + w^\dagger N^2 = \bar{\mathcal{B}}. \tag{43}$$

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We thus succeeded in deriving a mean buoyancy equation in which there is no explicit eddy term; buoyancy is transported solely through the mean vertical residual motion. It might be thought, of course, that the eddy terms are still there, implicit in $w^\dagger$. But this was also true of w_a which, as noted earlier, is in general influenced by the eddies. What we have done, is to redefine this influence, so as to put the mean buoyancy budget into its simplest possible form.

We can complete the transformed system of equations,

$$\partial_t \bar{u} - f_0 \bar{v}^\dagger = \bar{\mathcal{G}}_x + \nabla \cdot \boldsymbol{F}, \tag{44}$$

$$f_0 \partial_z \bar{u} = -\partial_y \bar{b}, \tag{45}$$

$$\partial_y \bar{v}^\dagger + \partial_z \bar{w}^\dagger = 0, \tag{46}$$

$$\partial_t \bar{b} + \bar{w}^\dagger N^2 = \bar{\mathcal{B}}, \tag{47}$$

where $\boldsymbol{F}$ is the Eliassen-Palm flux.

This transformation makes the role of eddies look quite different–even though the physics described by equations (44) through (47) is the same described by (34) through (37). The main advantage is that in terms of $\bar{v}^\dagger$, $\bar{w}^\dagger$, $\partial_t \bar{u}$, and $\partial_t \bar{b}$, the only term representing the eddy forcing is $\nabla \cdot \boldsymbol{F} = \overline{v'q'}$. This eddy forcing appears as an effective body force in the mean momentum equation. It is clear therefore that, under non-acceleration conditions (when $\nabla \cdot \boldsymbol{F} = 0$ and the boundary conditions are independent of eddy dependent terms), $\bar{v}^\dagger$, $\bar{w}^\dagger$, $\partial_t \bar{u}$, and $\partial_t \bar{b}$, are independent of the eddies.

When non-acceleration conditions are not satisfied, the transformed equations offer a more transparent approach to the eddy-mean flow interaction problem, simply because the single term represented by the effective force $\nabla \cdot \boldsymbol{F}$ entirely describes the eddy forcing of the mean state. In fact, this formulation gives us another interpretation of $\boldsymbol{F}$, as an eddy flux of transformed negative (easterly) momentum, which is a more reliable measure of eddy transport of momentum than $\overline{u'v'}$ itself.

The interpretation of $\boldsymbol{F}$ as a momentum flux may seem to be a result of mathematical tinkering. However, it should be remembered that the process of taking a mean is an arbitrary one–there is no unique way of doing it. Thus, it is legitimate to choose the definition of mean that simplifies the most the problem at hand. The Transformed

Eulerian Mean equations indeed give us a clearer picture on what is going on when eddies interact with a mean flow.

Parameterizing macroscale turbulence in numerical models

So far we avoided getting our hands dirty to find closures that relate the eddy fluxes to the mean flow. The TEM formalism is however often invoked to derive parameterizations of the interaction between large-scale mean flows and small-scale transient eddy motions. In this section, we will use the results of TEM together with some physical insight to derive sets of equations where the eddy terms do not appear explicitly. Two approaches are used in the literature, one based on an energetic argument, and the other on potential vorticity mixing theory.

The potential vorticity mixing argument

We have shown that in steady state for homogeneous turbulence, the eddy QGPV flux must be down its mean gradient,

$$\overline{v'q'} = -K\bar{q}_y. \tag{48}$$

Plugging this closure in the expression for the Eliassen-Palm fluxes gives,

$$\nabla \cdot \boldsymbol{F} = \overline{v'q'} = -K\left[\beta - \partial_{yy}\bar{u} + f_0\partial_z\left(\frac{\partial_y\bar{b}}{N^2}\right)\right]. \tag{49}$$

This expression for the eddy forcing of the residual circulation differs from that in (60). The two expressions are equivalent if 1) K is constant, 2) there is no planetary PV gradient ($\beta = 0$), and 3) PV is dominated by the baroclinic term. In the ocean condition 3 is often satisfied. Condition 2 is harder to assess, because it depends on whether eddies mix on distances large enough to feel the effect of β. Condition 1 instead cannot be satisfied, because one needs to impose $K = 0$ at the boundaries in the Gent-McWilliams parameterization and therefore K cannot be constant. Thus the two closure schemes are different. It is open to debate which approach is more appropriate. A good discussion can be found in the paper by Treguier et al. (1997).

Further reading:

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The energetic argument

The energetic argument has been used to derive eddy parameterizations in the ocean only. Thus we will restrict our scope to ocean dynamics in this section.

Although mesoscale eddy motions can be directly generated by external forcing, like the wind field, most of the mesoscale eddy energy is believed to be the result of instabilities in many forms (Pedlosky, 1987). The common belief is that eddies are generated by extracting kinetic and potential energy from the mean flow. This might not be the whole story though: in two dimensional and quasi-geostrophic turbulence, eddy motions can create an inverse energy cascade and return some of the energy back to the mean flow. The point here is that an analysis of the exchange of energy between mean and eddy motions might be fruitful to derive parameterizations.

The total mechanical energy is given by the sum of the kinetic K and potential energies P, which in the geostrophic approximation are,

$$K = \frac{1}{2}(u^2 + v^2), \qquad P = \frac{1}{2}\frac{b^2}{N^2}. \tag{50}$$

Conservation of total energy takes the form,

$$\left[\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right][K+P] + \frac{1}{\rho_0}\nabla\cdot(p\boldsymbol{u}_a) = \boldsymbol{u}\cdot\boldsymbol{\mathcal{G}} + \frac{b\mathcal{B}}{N^2}. \tag{51}$$

Exchange of energy between eddies and a zonal flow may be defined following the separation of the zonally averaged kinetic and potential energies into components associated with the eddy and mean motions. In the quasi-geostrophic approximation, this is straightforward,

$$K_M = \tfrac{1}{2}(\bar{u}^2 + \bar{v}^2), \qquad P_M = \frac{1}{2}\frac{\bar{b}^2}{N^2}, \tag{52}$$

$$K_E = \tfrac{1}{2}\left(\overline{u'^2} + \overline{v'^2}\right), \qquad P_E = \frac{1}{2}\frac{\overline{b'^2}}{N^2}. \tag{53}$$

Let us assume once again that the basic state is a zonal flow, *i.e.* $\bar{u} = \bar{u}(y, z, t)$, $\bar{b} = \bar{b}(y, z, t)$, but $\bar{v} = \bar{w} = 0$. The equations for the mean kinetic and potential energies are,

$$\frac{\partial K_M}{\partial t} + \frac{1}{\rho_0}\nabla \cdot (\bar{\boldsymbol{u}}_a \bar{p}) = \bar{b}\bar{w}_a - \bar{u}\partial_y(\overline{u'v'}) + \bar{u}\bar{\mathcal{G}}, \tag{54}$$

$$\frac{\partial P_M}{\partial t} + \bar{b}\bar{w}_a = -\bar{b}\partial_y\left(\frac{\overline{v'b'}}{N^2}\right) + \frac{\bar{b}\bar{\mathcal{B}}}{N^2}. \tag{55}$$

The eddy terms on the rhs represent conversion of mean energy into turbulent energy and are often associated with instabilities of the mean flow.

Equations (54) and (55) can be combined together in the form,

$$\frac{\partial}{\partial t}(K_M + P_M) + \frac{1}{\rho_0}\nabla \cdot (\bar{\boldsymbol{u}}_a \bar{p}) =$$
$$= -\partial_y\left(\bar{u}\overline{u'v'} + \bar{b}\frac{\overline{v'b'}}{N^2}\right) + \overline{u'v'}\partial_y\bar{u} + \frac{\overline{v'b'}}{N^2}\partial_y\bar{b} + \bar{u}\bar{\mathcal{G}} + \frac{\bar{b}\bar{\mathcal{B}}}{N^2}. \tag{56}$$

The ocean is a strongly stratified fluid and most of the energy in the basic state is stored as potential energy due to tilted isopycnal surfaces. This energy is converted into mesoscale eddy motions mainly through baroclinic instabilities. Thus in equation (56) the buoyancy eddy flux terms typically dominate over the eddy momentum flux terms.

The divergent terms represent transport of eddy activity from one region to another. In a statistically steady state, we can assume that there is no net transport of mean mechanical energy out of the domain considered. Neglecting the kinetic energy loss terms and the divergent terms, we have that on average,

$$\frac{\partial}{\partial t}(K_M + P_M) \sim \frac{\overline{v'b'}}{N^2}\partial_y\bar{b} + \text{external forcing}. \tag{57}$$

Baroclinic instability tends to extract potential energy from the mean state. The simplest closure that would ensure that energy is always released from the mean state and lost into the eddy filed is,

$$\overline{v'b'} = -K\partial_y\bar{b}. \tag{58}$$

This closure scheme was first proposed by Gent and McWilliams in 1990, and it is now in use in most coarse-resolution ocean models.

In terms of the TEM, this closure provide an estimate of the eddy induced circulation,

$$\chi_c = -K\frac{\partial_y\bar{b}}{N^2} \tag{59}$$

The parameterization of Gent and McWilliams is thus equivalent to assuming that the eddy induced circulation is proportional to the isopycnal slope. As long as mean isopycnals are tilted, there is available potential energy to drive an eddy-induced circulation.

In terms of the Transformed eulerian mean formalism the parameterization of Gent and McWilliams is as a closure for the eddy forcing of the residual circulation, *i.e.*,

$$\nabla \cdot \boldsymbol{F} \approx -f_0 \partial_z \left[K \frac{\partial_y \bar{b}}{N^2} \right]. \tag{60}$$

In this closure the eddy stress is proportional to the isopycnal slope. In order to satisfy conservation of mean momentum, it is customary to impose $K = 0$ at the ocean surface and ocean bottom.

Further reading

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