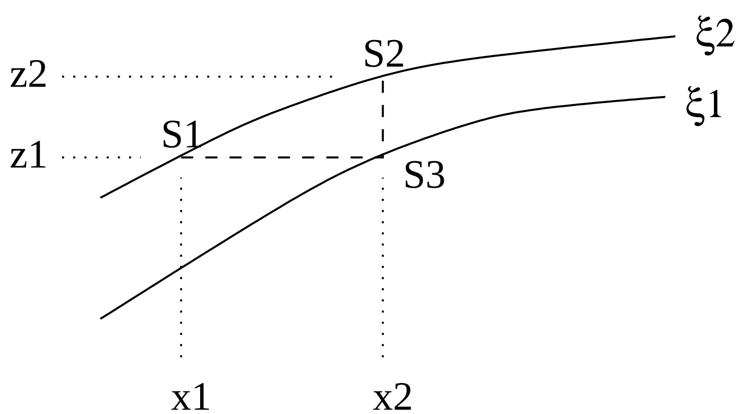
Change of Coordinates (non–orthogonal)

Different vertical coordinates

Suppose we have a property $S(x, y, z, t)$ and want to express it as $S(x, y, \xi, t)$ in terms of a different vertical coordinate $\xi = \xi(x, y, z, t)$ — e.g., pressure, so that we look at the temperature vs. latitude and longitude on the 500mb surface or the 750mb surface. What is the relationship between derivatives like the rate of change with x along a horizontal line $\left(\frac{\partial S}{\partial x}\right)_z$ and the rate of change with horizontal distance along a constant ξ surface $\left(\frac{\partial S}{\partial x}\right)_\xi$?

Let us look at this graphically:



The derivatives in question are

$$\left(\frac{\partial S}{\partial x}\right)_z = \frac{S_3 - S_1}{x_2 - x_1} \quad , \quad \left(\frac{\partial S}{\partial x}\right)_\xi = \frac{S_2 - S_1}{x_2 - x_1}$$

We can relate these two by using the vertical changes

$$S_2 - S_3 = \left(\frac{\partial S}{\partial z}\right)_x (z_2 - z_1) = \left(\frac{\partial S}{\partial \xi}\right)_x (\xi_2 - \xi_1)$$

Using this to eliminate S_3 from the rate of change along a horizontal surface gives

$$\begin{aligned}
\left(\frac{\partial S}{\partial x}\right)_z &= \frac{S_3 - S_2}{x_2 - x_1} + \frac{S_2 - S_1}{x_2 - x_1} \\
&= \frac{S_2 - S_1}{x_2 - x_1} - \left(\frac{\partial S}{\partial \xi}\right)_x \frac{\xi_2 - \xi_1}{x_2 - x_1} \\
&= \left(\frac{\partial S}{\partial x}\right)_\xi - \left(\frac{\partial S}{\partial \xi}\right)_x \frac{z_2 - z_1}{x_2 - x_1} \frac{\xi_2 - \xi_1}{z_2 - z_1} \\
&= \left(\frac{\partial S}{\partial x}\right)_\xi - \left(\frac{\partial S}{\partial \xi}\right)_x \left(\frac{\partial z}{\partial x}\right)_\xi \Big/ \left(\frac{\partial z}{\partial \xi}\right)_x
\end{aligned}$$

Likewise

$$\left(\frac{\partial S}{\partial z}\right)_x = \left(\frac{\partial S}{\partial \xi}\right)_x \Big/ \left(\frac{\partial z}{\partial \xi}\right)_x$$

Thus, to change coordinates we replace $\frac{\partial S}{\partial x}$ by

$$\left(\frac{\partial S}{\partial x}\right)_z \rightarrow \left(\frac{\partial S}{\partial x}\right)_\xi - \left(\frac{\partial S}{\partial \xi}\right)_x \frac{\left(\frac{\partial z}{\partial x}\right)_\xi}{\left(\frac{\partial z}{\partial \xi}\right)_x}$$

with similar forms for $\frac{\partial S}{\partial y}$ and $\frac{\partial S}{\partial t}$; the vertical replacement is

$$\left(\frac{\partial S}{\partial z}\right)_x \rightarrow \frac{\left(\frac{\partial S}{\partial \xi}\right)_x}{\left(\frac{\partial z}{\partial \xi}\right)_x}$$

Math note: General coordinate change

There is a fairly straightforward mathematical procedure for changing coordinates from one system to another, even if the second is not orthogonal. Suppose we have a function $S(\mathbf{x})$ and wish to express it and its derivatives as functions of the new coordinates $\boldsymbol{\xi}$. We could use the chain rule to find

$$\frac{\partial S}{\partial x_i} = \frac{\partial \xi_j}{\partial x_i}\frac{\partial S}{\partial \xi_j} \tag{1}$$

But this may not be adequate, for the following reason. We wish to have coefficients in the final equations expressed as functions of the new coordinates; however, quantities such as

$$\frac{\partial \xi_1}{\partial x_3}$$

are more likely to be known as functions of $\mathbf{x}$.

To accomplish the goal of having all terms expressed in the new coordinates, we begin with the opposite form

$$\frac{\partial S}{\partial \xi_i} = \frac{\partial x_j}{\partial \xi_i}\frac{\partial S}{\partial x_j} \quad \mathit{or} \quad \nabla_x S = \mathbf{T}\nabla_\xi S \tag{2}$$

and assume that the $\frac{\partial x_j}{\partial \xi_i}$ terms are functions of $\boldsymbol{\xi}$. We can express derivatives in the old coordinate system in terms of derivates in the new system by inverting the transformation matrix:

$$\frac{\partial S}{\partial x_i} = \left[\frac{\partial x_i}{\partial \xi_j}\right]^{-1}\frac{\partial S}{\partial \xi_j} \quad \mathit{or} \quad \nabla_\xi S = \mathbf{T}^{-1}\nabla_x S \tag{3}$$

In terms of the Jacobian matrix

$$\frac{\partial(A,B,C)}{\partial(\xi_1,\xi_2,\xi_3)} \equiv det \begin{pmatrix} \frac{\partial A}{\partial \xi_1} & \frac{\partial A}{\partial \xi_2} & \frac{\partial A}{\partial \xi_3} \\ \frac{\partial B}{\partial \xi_1} & \frac{\partial B}{\partial \xi_2} & \frac{\partial B}{\partial \xi_3} \\ \frac{\partial C}{\partial \xi_1} & \frac{\partial C}{\partial \xi_2} & \frac{\partial C}{\partial \xi_3} \end{pmatrix}$$

we have

$$\frac{\partial S}{\partial x_1} = \frac{\partial(S,x_2,x_3)}{\partial(x_1,x_2,x_3)} = \frac{\partial(S,x_2,x_3)}{\partial(\xi_1,\xi_2,\xi_3)} \Big/ \frac{\partial(x_1,x_2,x_3)}{\partial(\xi_1,\xi_2,\xi_3)}$$

etc.

Example

If we take polar coordinates as a specific case, we have the relationship between the old and new coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z'$$

So that the transformation matrix matrix $Tij = \frac{\partial x_j}{\partial \xi_i}$ in (2) is

$$\mathbf{T} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The inverse is

$$\mathbf{T}^{-1} = \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta & 0 \\ \sin\theta & \frac{1}{r}\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that

$$\psi_x = \cos\theta\ \psi_r - \frac{1}{r}\sin\theta\ \psi_\theta$$
$$\psi_y = \sin\theta\ \psi_r + \frac{1}{r}\cos\theta\ \psi_\theta$$
$$\psi_z = \psi_{z'}$$

using subscript notation for derivatives.

Change in vertical coordinate

If we switch from $x,\ y,\ z$ to $x',\ y',\ \xi$, the transformation matrix is

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & \dfrac{\partial z}{\partial x'} \\ 0 & 1 & \dfrac{\partial z}{\partial y'} \\ 0 & 0 & \dfrac{\partial z}{\partial \xi} \end{pmatrix}$$

and its inverse is

$$\mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 & -\dfrac{\partial z}{\partial x'} \Big/ \dfrac{\partial z}{\partial \xi} \\ 0 & 1 & -\dfrac{\partial z}{\partial y'} \Big/ \dfrac{\partial z}{\partial \xi} \\ 0 & 0 & 1 \Big/ \dfrac{\partial z}{\partial \xi} \end{pmatrix}$$

Thus we can replace horizontal gradients

$$\nabla \longrightarrow \nabla - \frac{\nabla z}{z_\xi} \frac{\partial}{\partial \xi}$$

vertical derivatives

$$\frac{\partial}{\partial z} \longrightarrow \frac{1}{z_\xi} \frac{\partial}{\partial \xi}$$

and time derivatives

$$\frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t} - \frac{z_t}{z_\xi} \frac{\partial}{\partial \xi}$$

in our original equations.

First, we note that the material derivative becomes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla + \frac{1}{z_\xi} (w - z_t - \mathbf{u} \cdot \nabla z) \frac{\partial}{\partial \xi}$$

and we can define the "vertical" velocity ω as

$$\omega = \frac{1}{z_\xi} (w - z_t - \mathbf{u} \cdot \nabla z)$$

so that the material derivative becomes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla + \omega \frac{\partial}{\partial \xi}$$

With this definition, we note that $w = \frac{D}{Dt} z$ as we might expect.

Transformed equations

The horizontal momentum equations become

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\varphi \qquad (e.1)$$

with $\varphi = gz$ being the geopotential; the hydrostatic balance is

$$\frac{\partial}{\partial\xi}\varphi = -\frac{1}{\rho}\frac{\partial}{\partial\xi}p \qquad (e.2a)$$

while the conservation of mass gives

$$\frac{1}{\rho}\frac{D}{Dt}\rho + \nabla\cdot\mathbf{u} - \frac{1}{z_\xi}\mathbf{u}_\xi\cdot\nabla z + \frac{1}{z_\xi}\frac{\partial}{\partial\xi}(\frac{D}{Dt}z) = 0$$

implying

$$\frac{1}{\rho}\frac{D}{Dt}\rho + \frac{1}{z_\xi}\frac{D}{Dt}z_\xi + \nabla\cdot\mathbf{u} + \frac{\partial}{\partial\xi}\omega$$

or

$$\frac{1}{p_\xi}\frac{D}{Dt}p_\xi + \nabla\cdot\mathbf{u} + \frac{\partial}{\partial\xi}\omega = 0 \qquad (e.3)$$

or

$$\frac{\partial}{\partial t}h + \nabla\cdot(h\mathbf{u}) = 0 \quad with \quad h = const. * p_\xi$$

Finally, the thermodynamic equation becomes

$$\frac{D}{Dt}\rho - \frac{1}{c_s^2}\frac{D}{Dt}p = 0 \qquad (e.4a)$$

in general. The potential vorticity (with η being the entropy) is

$$q = -\frac{g}{p_\xi}(\nabla_3 \times \mathbf{u} + f\hat{\mathbf{k}})\cdot\nabla_3\eta \qquad (e.5)$$

with the ∇_3 notation indicating the vertical derivatives are included.

Vertical coordinate function of pressure

When the vertical coordinate is a function of pressure $\xi = \xi(p)$ or $p = p(\xi)$, we can define $p_\xi \equiv -g\rho_c(\xi)$ and simplify the hydrostatic equation to

$$\frac{\partial}{\partial \xi}\varphi = g\frac{\rho_c}{\rho} \equiv b + g$$

We can replace $\varphi = \varphi' + g\xi$ to get

$$\frac{\partial}{\partial \xi}\varphi' = b$$

The equations become (dropping the $'$ in φ)

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\varphi \qquad (p.1)$$

$$\frac{\partial}{\partial \xi}\varphi = b \qquad (p.2)$$

$$\nabla \cdot \mathbf{u} + \frac{1}{\rho_c}\frac{\partial}{\partial \xi}(\rho_c\omega) = 0 \qquad (p.3)$$

$$\frac{D}{Dt}\rho + \omega\frac{g\rho_c}{c_s^2} = 0 \quad or \quad \frac{D}{Dt}b + \omega\left[-g\frac{\rho_{c\xi}}{\rho} - \frac{g^2\rho_c^2}{\rho^2 c_s^2}\right] = 0$$

The last equation can also be written

$$\frac{\partial}{\partial t}b + \mathbf{u}\cdot\nabla b + \omega\left[b_\xi - (g+b)\frac{\rho_{c\xi}}{\rho_c} - \frac{(g+b)^2}{c_s^2}\right] = 0$$

or

$$\frac{\partial}{\partial t}b + \mathbf{u}\cdot\nabla b + \omega\mathcal{S} = 0 \qquad (p.4)$$

with the stratification parameter $\mathcal{S}$

$$\mathcal{S} \equiv \frac{\rho_c^2}{\rho^2}N^2 = b_\xi - (g+b)\frac{\rho_{c\xi}}{\rho_c} - \frac{(g+b)^2}{c_s^2} \qquad (p.5a)$$

defined in terms of the Brunt-Väisälä frequency

$$N^2 = -g\frac{1}{\rho}\frac{\partial}{\partial z}\rho - \frac{g^2}{c_s^2} = -g\frac{\rho_\xi}{\rho_c} - \frac{g^2}{c_s^2} = \frac{g^2}{(g+b)^2}b_\xi - \frac{g^2}{(g+b)}\frac{\rho_{c\xi}}{\rho_c} - \frac{g^2}{c_s^2} \qquad (p.5b)$$

The PV is

$$q = \frac{1}{\rho_c}(\nabla_3 \times \mathbf{u} + f\hat{\mathbf{k}})\cdot\nabla_3\eta \qquad (p.6)$$

We shall use eqns. p1-p4 as our basic set.

The boundary conditions are a bit tricky; if the bottom is at $z = h(x, y)$, we get an implicit equation for the surface pressure $\xi_s(x, y, t)$:

$$\varphi(x, y, \xi_s(x, y, t), t) + g\xi_s = gh(x, y) \tag{b.1}$$

We also have the kinematic condition

$$w\left(= \frac{D}{Dt}z\right) = \frac{D}{Dt}h \quad \Rightarrow \quad \frac{D}{Dt}(\varphi - gh) = 0$$

Together, these two imply

$$\omega = \frac{D}{Dt}\xi_s \quad at \quad \xi = \xi_s \tag{b.2}$$

Thermodynamics

For an ideal gas, we can simplify the thermodynamics using $\eta = c_p \ln \theta$

$$\frac{D}{Dt}\theta = 0 \tag{p.7}$$

with the potential temperature being

$$\theta = \theta_0 \frac{\rho_0}{\rho}\left(\frac{p}{p_0}\right)^{1/\gamma}$$

($\gamma = c_p/c_v$). Thus, the buoyancy becomes

$$b = g\frac{\rho_c}{\rho_0}\left(\frac{p}{p_0}\right)^{-1/\gamma}\frac{\theta}{\theta_0} - g \equiv G(\xi)\frac{\theta}{\theta_0} \tag{p.8}$$

With a little work, you can substitute (p.8) into (p.4), using $c_s^2 = \gamma p/\rho$, to show that (p.7) holds. The Brunt-Väisälä frequency is

$$N^2 = g\frac{\partial}{\partial z}\ln\theta = g\frac{\rho}{\rho_c}\frac{\partial}{\partial \xi}\ln\theta \quad , \quad \mathcal{S} = g\frac{\rho_c}{\rho}\frac{\partial}{\partial \xi}\ln\theta$$

Linearized equations

The wave equations for this system are

$$\frac{\partial}{\partial t}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\phi'$$

$$\frac{\partial}{\partial \xi}\phi' = b'$$

$$\nabla \cdot \mathbf{u} + \frac{1}{\rho_c}\frac{\partial}{\partial \xi}(\rho_c \omega) = 0$$

$$\frac{\partial}{\partial t}b' + \omega\overline{\mathcal{S}} = 0$$

If we make the particular choice of $\rho_c = \overline{\rho}$, so that ξ is just the height in the resting atmosphere, we have $\overline{b} = g$, $\overline{\mathcal{S}} = \overline{N^2}$, and the equations look like the Boussinesq form except for the $\overline{\rho}$ factors in the stretching term. We can separate variables

$$\mathbf{u} \to \mathbf{u}(\mathbf{x}, t)F(z) \quad , \quad \phi' \to \phi'(\mathbf{x}, t)F(z) \quad , \quad b' \to b'(\mathbf{x}, t)\frac{\partial F}{\partial \xi} \quad , \quad \omega = -\frac{\partial \phi'}{\partial t}\frac{1}{\overline{N^2}}\frac{\partial F}{\partial \xi}$$

The mass conservation equation gives

$$\frac{\partial \phi'}{\partial t}\left[-\frac{1}{\overline{\rho}}\frac{\partial}{\partial \xi}\frac{\overline{\rho}}{\overline{N^2}}\frac{\partial}{\partial \xi}F\right] + \nabla \cdot \mathbf{u}F = 0$$

giving again the vertical structure eigenvalue equation

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial \xi}\frac{\overline{\rho}}{\overline{N^2}}\frac{\partial}{\partial \xi}F = -\frac{1}{gH_e}F$$

and the horizontal equations

$$\frac{\partial}{\partial t}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\phi'$$

$$\frac{1}{gH_e}\frac{\partial \phi'}{\partial t} + \nabla \cdot \mathbf{u} = 0$$

The lower boundary condition gives the surface pressure

$$\frac{\partial \overline{\phi}}{\partial \xi}\xi_s + \phi'(\mathbf{x}, 0, t) \simeq 0 \quad \Rightarrow \quad \xi_s = -\frac{1}{g}\phi'(\mathbf{x}, t)F(0)$$

and its evolution

$$\frac{\partial \xi_s}{\partial t} = \omega(\mathbf{x}, 0, t) = -\frac{\partial \phi'}{\partial t}\frac{1}{\overline{N^2}}\frac{\partial F}{\partial \xi} \quad \Rightarrow \quad \frac{\partial F}{\partial \xi} = \frac{\overline{N^2}}{g}F \quad at \ \ \xi = 0$$

Often, however, the simpler condition $\omega = 0 \ \Rightarrow \ \frac{\partial F}{\partial \xi} = 0$ is used.

Isothermal atmosphere

One case that can be worked out completely is the isothermal basic state. Using the gas law gives $\overline{p} = \overline{\rho} RT$; the hydrostatic equation then gives

$$\overline{p} = p_0 \exp(-z/H_s) \quad , \quad \overline{\rho} = \rho_0 \exp(-z/H_s) \quad , \quad H_s = RT/g \quad , \quad p_0 = \rho_0 g H_s$$

— the density decays exponentially with a scale height H_s. We can just choose $\xi = H_s \ln(p_0/p)$ so that it's the same as height. The associated density $\rho_c = \overline{\rho}$ as before. When we calculate the Brunt-Väisälä frequency, we get

$$\overline{N^2} = \frac{g}{H_s} - \frac{g^2}{c_s^2} = \frac{g}{H_s}\left[1 - \frac{c_v}{c_p}\right] = \frac{g}{H_s}\frac{R}{c_p}$$

and discover that it is constant. The vertical structure equation becomes

$$\frac{\partial^2 F}{\partial \xi^2} - \frac{1}{H_s}\frac{\partial F}{\partial \xi} = -\frac{1}{H_s H_e}\frac{R}{c_p} F$$

Therefore F will have exponential solutions

$$F = \exp(\alpha z/H_s) \quad , \quad \alpha^2 - \alpha + \frac{H_s}{H_e}\frac{R}{c_p} = 0 \quad , \quad \alpha = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\frac{R}{c_p}\frac{H_s}{H_e}}$$

If we start with the case when the argument of the square root is positive, we must eliminate the large root, since it has an energy density $\overline{\rho} u^2 \sim \exp([2\alpha - 1]\xi/H_s)$ which grows towards infinity. Therefore we can only accept the negative sign, giving

$$F = \exp\left(\left[1 - \sqrt{1 - 4\frac{R}{c_p}\frac{H_s}{H_e}}\right]\frac{\xi}{2H_s}\right)$$

The lower boundary condition gives (for $\omega = 0$)

$$\alpha = 0 \quad \Rightarrow \quad \frac{1}{gH_e} \to 0 \quad , \quad F = 1$$

or for the full condition

$$\alpha = \frac{H_s \overline{N^2}}{g} = \frac{R}{c_p} \quad \Rightarrow \quad H_e = \frac{H_s}{1 - \frac{R}{c_p}} = \frac{c_p}{c_v} H_s \quad , \quad F = \exp(\frac{R}{c_p}\frac{\xi}{H_s})$$

which will be well-behaved as long as $c_p > 2R$ (for the atmosphere $c_v,\ c_p,\ R = 718,\ 1005,\ 287.1\ J/kg/K^o$ (Tsonis, *An Introduction to Atmospheric Thermodynamics*) so that this condition is fine. The equivalent depth is 40% larger than the scale height. Over one scale height, F grows by a factor of $\exp(R/c_p) = 1.33$ while the kinetic energy density decreases by $\exp(2\frac{R}{c_p} - 1) = 0.65$. This is called the *equivalent barotropic* mode.

Are there any other modes? The derivation above makes it clear that this is the only mode with $H_e > 4(R/c_p)H_s = 1.14H_s$. What about the modes with complex α which have energies remaining order one at infinity? The lower boundary condition clearly requires both the α_+ and α_- modes; however, the latter will have downward energy flux. To maintain such a mode, we require a reflecting surface or an energy source high in the atmosphere. This will not happen for a resting atmosphere; therefore, the only mode available is the equivalent barotropic mode.

Special cases of the pressure-like equations

Pressure coords

In the atmosphere, the standard choice is pressure coordinates $\xi = p$

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\varphi + G(p)\frac{\theta}{\theta_0}\,\hat{\mathbf{k}}$$
$$\nabla \cdot \mathbf{u} + \omega_p = 0$$
$$\frac{D}{Dt}\theta = 0$$
$$G = -\frac{R\theta_0}{p_0}\left(\frac{p}{p_0}\right)^{-1/\gamma}$$

or

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\varphi + b\,\hat{\mathbf{k}}$$
$$\nabla \cdot \mathbf{u} + \omega_p = 0 \quad \textit{with} \quad b = G(p)\frac{\theta}{\theta_0}$$
$$\frac{D}{Dt}b + \omega b\frac{1}{\gamma p} = 0$$

Log p

But the log form is also convenient especially if we work with a near-isothermal stratification so that $\overline{\rho} = \rho_c = \rho_0 \exp(-\xi/H)$

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\varphi + g(\theta/\overline{\theta})\,\hat{\mathbf{k}}$$
$$\nabla \cdot \mathbf{u} + (\frac{\partial}{\partial\xi} - \frac{1}{H})\omega = 0$$
$$\frac{D}{Dt}\theta = 0$$
$$q = \frac{1}{\rho}(\nabla_3 \times \mathbf{u} + f\hat{\mathbf{k}}) \cdot \nabla_3 \ln\theta$$
$$Q = \nabla^2\psi + \frac{f^2}{\overline{N^2}}(\frac{\partial}{\partial\xi} - \frac{1}{H})\frac{\partial}{\partial\xi}\psi + f$$
$$\overline{\rho} = \rho_0 e^{-\xi/H} \quad , \quad \overline{N^2} = \frac{g}{H}\frac{\gamma - 1}{\gamma} \quad , \quad \overline{\theta} = \theta_0 e^{(\gamma-1)\xi/\gamma H}$$

For both systems, the choice with ξ corresponding to the resting atmosphere so that $\rho_c = \overline{\rho}$ gives a quasi-Boussinesq model

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\varphi + b\hat{\mathbf{k}}$$

$$\nabla \cdot \mathbf{u} + \frac{1}{\overline{\rho}}\frac{\partial}{\partial\xi}(\overline{\rho}\omega) = 0$$

$$\frac{\partial}{\partial t}b + \mathbf{u} \cdot \nabla b + \omega\mathcal{S} = 0$$

$$q = \frac{1}{\rho}(\nabla_3 \times \mathbf{u} + f\hat{\mathbf{k}}) \cdot \nabla_3\eta$$

$$Q = \nabla^2\psi + \frac{1}{\overline{\rho}}\frac{\partial}{\partial\xi}\overline{\rho}\frac{f^2}{\overline{N^2}}\frac{\partial}{\partial\xi}\psi + f$$

$$\overline{\mathcal{S}} = \overline{N^2} = -g\frac{\overline{\rho}_\xi}{\overline{\rho}} - \frac{g^2}{\overline{c}_s^2} \quad , \quad b = g\frac{\theta}{\overline{\theta}} \quad , \quad \overline{N^2} = g\frac{\partial}{\partial\xi}\ln\overline{\theta}$$

For the ocean, we usually use $(p_0 - p)/\rho_0 g$ and ignore the difference between $\mathcal{S}$ and N^2, giving a Boussinesq form

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\varphi + \tilde{b}\hat{\mathbf{k}}$$

$$\nabla \cdot \mathbf{u} + \omega_\xi = 0$$

$$\frac{D}{Dt}\tilde{b} = 0$$

$$q = \frac{1}{\rho_0}(\nabla_3 \times \mathbf{u} + f\hat{\mathbf{k}}) \cdot \nabla_3 b$$

$$Q = \nabla^2\psi + \frac{\partial}{\partial\xi}\frac{f^2}{\overline{N^2}}\frac{\partial}{\partial\xi}\psi + f$$

$$\tilde{b} = b - \frac{g^2}{c_s^2}\xi \quad , \quad \mathcal{S} \simeq b_\xi - \frac{g^2}{c_s^2}$$

Summary table

ξ	ρ_c	G	$\overline{\mathcal{S}}$(atm.),$\overline{\mathcal{S}}$(oc.)
p	$-1/g$	$-\frac{R}{p_0}\left(\frac{\xi}{p_0}\right)^{-1/\gamma}$	$-\frac{1}{\overline{\rho}}\frac{\partial}{\partial\xi}\ln\overline{\theta}$
$(p_0-p)/\rho_0 g$	ρ_0	$\frac{g}{\theta_0}(1-\frac{\xi}{H})^{-1/\gamma}$	$g\frac{\rho_0}{\overline{\rho}}\frac{\partial}{\partial\xi}\ln\overline{\theta}$ $\frac{\rho_0^2}{\overline{\rho}^2}\overline{N^2}\simeq\overline{N^2}$
$-H\ln\frac{p}{p_0}$	$\rho_0 e^{-\xi/H}$	$\frac{g}{\theta_0}\exp(-\frac{\gamma-1}{\gamma}\frac{\xi}{H})$	$g\frac{\rho_0 e^{-\xi/H}}{\overline{\rho}}\frac{\partial}{\partial\xi}\ln\overline{\theta}$
$\frac{H\gamma}{\gamma-1}\left[1-\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right]$	$\rho_0\left[1-\frac{\xi}{H}\frac{\gamma-1}{\gamma}\right]^{1/(\gamma-1)}$	$\frac{g}{\theta_0}$	$g\frac{\rho_0}{\overline{\rho}}\left[1-\frac{\xi}{H}\frac{\gamma-1}{\gamma}\right]^{\frac{1}{(\gamma-1)}}\frac{\partial}{\partial\xi}\ln\overline{\theta}$
$-\int_{p_0}^{p}dp'\frac{p'}{\overline{\rho}(p')g}$	$\overline{\rho}$	$\frac{g}{\overline{\theta}}$	$g\frac{\partial}{\partial\xi}\ln\overline{\theta}$ $\overline{N^2}$

In this chart, p_0 and ρ_0 are reference values; the scale height is related to these two by $gH = RT_0 = R\theta_0 = p_0/\rho_0$.