Change of Coordinates (non–orthogonal)

Different vertical coordinates

Suppose we have a property \( S(x, y, z, t) \) and want to express it as \( S(x, y, \xi, t) \) in terms of a different vertical coordinate \( \xi = \xi(x, y, z, t) \) — e.g., pressure, so that we look at the temperature vs. latitude and longitude on the 500mb surface or the 750mb surface. What is the relationship between derivatives like the rate of change with \( x \) along a horizontal line \( \frac{\partial S}{\partial x} \) and the rate of change with horizontal distance along a constant \( \xi \) surface \( \frac{\partial S}{\partial \xi} \)?

Let us look at this graphically:

The derivatives in question are

\[
\left( \frac{\partial S}{\partial x} \right)_z = \frac{S_3 - S_1}{x_2 - x_1}, \quad \left( \frac{\partial S}{\partial x} \right)_\xi = \frac{S_2 - S_1}{x_2 - x_1}
\]

We can relate these two by using the vertical changes

\[
S_2 - S_3 = \left( \frac{\partial S}{\partial z} \right)_x (z_2 - z_1) = \left( \frac{\partial S}{\partial \xi} \right)_x (\xi_2 - \xi_1)
\]

Using this to eliminate \( S_3 \) from the rate of change along a horizontal surface gives

\[
\left( \frac{\partial S}{\partial x} \right)_z = \frac{S_3 - S_2}{x_2 - x_1} + \frac{S_2 - S_1}{x_2 - x_1}
\]

\[
= \frac{S_2 - S_1}{x_2 - x_1} - \left( \frac{\partial S}{\partial \xi} \right)_x \frac{\xi_2 - \xi_1}{x_2 - x_1}
\]

\[
= \frac{\partial S}{\partial x} \xi - \left( \frac{\partial S}{\partial \xi} \right)_x \frac{z_2 - z_1}{x_2 - x_1} \frac{\xi_2 - \xi_1}{z_2 - z_1}
\]

\[
= \frac{\partial S}{\partial \xi} \frac{\partial S}{\partial x} \xi - \left( \frac{\partial S}{\partial \xi} \right)_x \frac{z_2 - z_1}{x_2 - x_1} \frac{\partial \xi}{\partial x} \frac{\partial z}{\partial x}
\]

\[
\frac{\partial S}{\partial x} \xi = \frac{\partial S}{\partial \xi} \frac{\partial S}{\partial x} \xi - \left( \frac{\partial S}{\partial \xi} \right)_x \frac{z_2 - z_1}{x_2 - x_1} \frac{\partial \xi}{\partial x} \frac{\partial z}{\partial x}
\]
Likewise

\[
\left( \frac{\partial S}{\partial z} \right)_x = \left( \frac{\partial S}{\partial \xi} \right)_x / \left( \frac{\partial z}{\partial \xi} \right)_x
\]

Thus, to change coordinates we replace \( \frac{\partial S}{\partial x} \) by

\[
\left( \frac{\partial S}{\partial x} \right)_z \rightarrow \left( \frac{\partial S}{\partial x} \right)_\xi - \left( \frac{\partial S}{\partial \xi} \right)_x \left( \frac{\partial z}{\partial x} \right)_\xi
\]

with similar forms for \( \frac{\partial S}{\partial y} \) and \( \frac{\partial S}{\partial t} \); the vertical replacement is

\[
\left( \frac{\partial S}{\partial z} \right)_x \rightarrow \left( \frac{\partial S}{\partial \xi} \right)_x
\]

**Math note: General coordinate change**

There is a fairly straightforward mathematical procedure for changing coordinates from one system to another, even if the second is not orthogonal. Suppose we have a function \( S(\mathbf{x}) \) and wish to express it and its derivatives as functions of the new coordinates \( \mathbf{\xi} \). We could use the chain rule to find

\[
\frac{\partial S}{\partial x_i} = \frac{\partial S}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i}
\]  

(1)

But this may not be adequate, for the following reason. We wish to have coefficients in the final equations expressed as functions of the new coordinates; however, quantities such as

\[
\frac{\partial \xi_1}{\partial x_3}
\]

are more likely to be known as functions of \( \mathbf{x} \).

To accomplish the goal of having all terms expressed in the new coordinates, we begin with the opposite form

\[
\frac{\partial S}{\partial \xi_i} = \frac{\partial x_j}{\partial \xi_i} \frac{\partial S}{\partial x_j} \quad \text{or} \quad \nabla_\xi S = \mathbf{T} \nabla_x S
\]  

(2)

and assume that the \( \frac{\partial x_i}{\partial \xi_i} \) terms are functions of \( \mathbf{\xi} \). We can express derivatives in the old coordinate system in terms of derivates in the new system by inverting the transformation matrix:

\[
\frac{\partial S}{\partial x_i} = \left[ \frac{\partial x_i}{\partial \xi_j} \right]^{-1} \frac{\partial S}{\partial \xi_j} \quad \text{or} \quad \nabla_\xi S = \mathbf{T}^{-1} \nabla_x S
\]  

(3)
In terms of the Jacobian matrix
\[
\frac{\partial (A, B, C)}{\partial (\xi_1, \xi_2, \xi_3)} \equiv \det \begin{vmatrix}
\frac{\partial A}{\partial \xi_1} & \frac{\partial A}{\partial \xi_2} & \frac{\partial A}{\partial \xi_3} \\
\frac{\partial B}{\partial \xi_1} & \frac{\partial B}{\partial \xi_2} & \frac{\partial B}{\partial \xi_3} \\
\frac{\partial C}{\partial \xi_1} & \frac{\partial C}{\partial \xi_2} & \frac{\partial C}{\partial \xi_3}
\end{vmatrix}
\]
we have
\[
\frac{\partial S}{\partial x_1} = \frac{\partial (S, x_2, x_3)}{\partial (x_1, x_2, x_3)} = \frac{\partial (S, x_2, x_3)}{\partial (\xi_1, \xi_2, \xi_3)} \cdot \frac{\partial (x_1, x_2, x_3)}{\partial (\xi_1, \xi_2, \xi_3)}
\]

etc.

**Example**

If we take polar coordinates as a specific case, we have the relationship between the old and new coordinates
\[
x = r \cos \theta \\
y = r \sin \theta \\
z = z'
\]

So that the transformation matrix matrix \( T^{ij} = \frac{\partial x_j}{\partial \xi_i} \) in (2) is
\[
T = \begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-r \sin \theta & r \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

The inverse is
\[
T^{-1} = \begin{pmatrix}
\cos \theta & -\frac{1}{r} \sin \theta & 0 \\
\sin \theta & \frac{1}{r} \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

so that
\[
\psi_x = \cos \theta \psi_r - \frac{1}{r} \sin \theta \psi_\theta \\
\psi_y = \sin \theta \psi_r + \frac{1}{r} \cos \theta \psi_\theta \\
\psi_z = \psi_z'
\]

using subscript notation for derivatives.
Change in vertical coordinate

If we switch from $x, y, z$ to $x', y', \xi$, the transformation matrix is

$$T = \begin{pmatrix}
1 & 0 & \frac{\partial z}{\partial x'} \\
0 & 1 & \frac{\partial z}{\partial y'} \\
0 & 0 & \frac{\partial z}{\partial \xi}
\end{pmatrix}$$

and its inverse is

$$T^{-1} = \begin{pmatrix}
1 & 0 & \frac{-\partial z}{\partial x'} / \frac{\partial z}{\partial \xi} \\
0 & 1 & \frac{-\partial z}{\partial y'} / \frac{\partial z}{\partial \xi} \\
0 & 0 & \frac{1}{\partial z / \partial \xi}
\end{pmatrix}$$

Thus we can replace horizontal gradients

$$\nabla \rightarrow \nabla - \frac{\nabla z \partial}{z\xi \partial \xi}$$

vertical derivatives

$$\frac{\partial}{\partial z} \rightarrow \frac{1}{z\xi} \frac{\partial}{\partial \xi}$$

and time derivatives

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - \frac{z_t}{z\xi} \frac{\partial}{\partial \xi}$$

in our original equations.

First, we note that the material derivative becomes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla + \frac{1}{z\xi}(w - z_t - \mathbf{u} \cdot \nabla z) \frac{\partial}{\partial \xi}$$

and we can define the “vertical” velocity $\omega$ as

$$\omega = \frac{1}{z\xi}(w - z_t - \mathbf{u} \cdot \nabla z)$$

so that the material derivative becomes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla + \omega \frac{\partial}{\partial \xi}$$

With this definition, we note that $w = \frac{D}{Dt} z$ as we might expect.
**Transformed equations**

The horizontal momentum equations become

\[
\frac{D}{Dt} u + f \hat{k} \times u = -\frac{1}{\rho} \nabla p - \nabla \varphi \quad (e.1)
\]

with \( \varphi = gz \) being the geopotential; the hydrostatic balance is

\[
\frac{\partial}{\partial \xi} \varphi = -\frac{1}{\rho} \frac{\partial}{\partial \xi} p \quad (e.2a)
\]

while the conservation of mass gives

\[
\frac{1}{\rho} \frac{D}{Dt} \rho + \nabla \cdot u - \frac{1}{z_\xi} u_\xi \cdot \nabla z + \frac{1}{z_\xi} \frac{\partial}{\partial \xi} \left( \frac{D}{Dt} z \right) = 0
\]

implying

\[
\frac{1}{\rho} \frac{D}{Dt} \rho + \frac{1}{z_\xi} \frac{D}{Dt} z_\xi + \nabla \cdot u + \frac{\partial}{\partial \xi} \omega
\]

or

\[
\frac{1}{p_\xi} \frac{D}{Dt} p_\xi + \nabla \cdot u + \frac{\partial}{\partial \xi} \omega = 0 \quad (e.3)
\]

Finally, the thermodynamic equation becomes

\[
\frac{D}{Dt} \rho - \frac{1}{c_s^2} \frac{D}{Dt} p = 0 \quad (e.4a)
\]

in general. The potential vorticity (with \( \eta \) being the entropy) is

\[
q = -\frac{g}{p_\xi} (\nabla_3 \times u + f \hat{k}) \cdot \nabla_3 \eta \quad (e.5)
\]

with the \( \nabla_3 \) notation indicating the vertical derivatives are included.
Vertical coordinate function of pressure

When the vertical coordinate is a function of pressure \( \xi = \xi(p) \) or \( p = p(\xi) \), we can define \( p_\xi \equiv -g \rho_c(\xi) \) and simplify the equations to

\[
\frac{D}{Dt} u + f \hat{k} \times u = -\nabla \varphi
\]  

\[(p.1)\]

\[
\frac{\partial}{\partial \xi} \varphi = g \frac{\rho_c}{\rho} \equiv b
\]  

\[(p.2)\]

\[
\nabla \cdot u + \frac{1}{\rho_c} \frac{\partial}{\partial \xi}(\rho_c \omega) = 0
\]  

\[(p.3)\]

\[
\frac{D}{Dt} \rho + \omega \frac{g \rho_c}{c_s^2} = 0 \quad \text{or} \quad \frac{D}{Dt} b + \omega \left[ -\frac{g \rho_c}{\rho} - \frac{g^2 \rho_c^2}{\rho^2 c_s^2} \right] = 0
\]

The last equation can also be written

\[
\frac{\partial}{\partial t} b + u \cdot \nabla b + \omega \left[ b_\xi - b \frac{\rho_c}{\rho_c} - \frac{b^2}{c_s^2} \right] = 0
\]  

or

\[
\frac{\partial}{\partial t} b + u \cdot \nabla b + \omega S = 0
\]  

\[(p.4)\]

with the stratification parameter \( S \)

\[
S \equiv \frac{\rho_c^2}{\rho_s^2} N^2 = b_\xi - b \frac{\rho_c}{\rho_c} - \frac{b^2}{c_s^2}
\]  

\[(p.5a)\]

defined in terms of the Brunt-Väisälä frequency

\[
N^2 = -\frac{1}{\rho} \frac{\partial}{\partial z} \rho - \frac{g^2}{c_s^2} = -g \frac{\rho_c}{\rho} - \frac{g^2}{c_s^2} = \frac{g^2}{b^2} b_\xi - \frac{g^2 \rho_c}{b} - \frac{g^2}{c_s^2}
\]  

\[(p.5b)\]

The PV is

\[
q = \frac{1}{\rho_c} \left( \nabla_3 \times u + f \hat{k} \right) \cdot \nabla_3 \eta
\]  

\[(p.6)\]

We shall use eqns. p1-p4 as our basic set.

The boundary conditions are a bit tricky; if the bottom is at \( z = h(x, y) \), we get an implicit equation for the surface pressure \( \xi_s(x, y, t) \):

\[
\phi(x, y, \xi_s(x, y, t), t) = gh(x, y)
\]  

\[(b.1)\]

We also have the kinematic condition

\[
w \left( \frac{D}{Dt} z \right) = \frac{D}{Dt} h \quad \Rightarrow \quad \frac{D}{Dt} (\phi - gh) = 0
\]

Together, these two imply

\[
\omega = \frac{D}{Dt} \xi \quad \text{at} \quad \xi = \xi_s
\]  

\[(b.2)\]
Thermodynamics

For an ideal gas, we can simplify the thermodynamics using $\eta = c_p \ln \theta$

$$\frac{D}{Dt} \theta = 0 \quad (p.7)$$

with the potential temperature being

$$\theta = \theta_0 \frac{\rho_0}{\rho} \left( \frac{p}{p_0} \right)^{1/\gamma}$$

$(\gamma = c_p/c_v)$. Thus, the buoyancy becomes

$$b = g \frac{\rho_c}{\rho_0} \left( \frac{p}{p_0} \right)^{-1/\gamma} \frac{\theta}{\theta_0} \equiv G(\xi) \theta \quad (p.8)$$

With a little work, you can substitute (p.8) into (p.4), using $c_s^2 = \gamma p/\rho$, to show that (p.7) holds. The Brunt-Väisälä frequency is

$$N^2 = g \frac{\partial}{\partial z} \ln \theta = g \frac{\rho}{\rho_c} \frac{\partial}{\partial \xi} \ln \theta , \quad \mathcal{S} = g \frac{\rho_c}{\rho} \frac{\partial}{\partial \xi} \ln \theta$$

Linearized equations

The wave equations for this system are

$$\frac{\partial}{\partial t} \mathbf{u} + f \mathbf{k} \times \mathbf{u} = -\nabla \phi'$$

$$\frac{\partial}{\partial \xi} \phi' = b'$$

$$\nabla \cdot \mathbf{u} + \frac{1}{\rho_c} \frac{\partial}{\partial \xi} (\rho_c \omega) = 0$$

$$\frac{\partial}{\partial t} b' + \omega \mathcal{S} = 0$$

If we make the particular choice of $\rho_c = \bar{\rho}$, so that $\xi$ is just the height in the resting atmosphere, we have $\bar{b} = g$, $\mathcal{S} = N^2$, and the equations look like the Boussinesq form except for the $\bar{\rho}$ factors in the stretching term. We can separate variables

$$\mathbf{u} \rightarrow \mathbf{u}(\mathbf{x}, t) F(z) , \quad \phi' \rightarrow \phi'(\mathbf{x}, t) F(z) , \quad b' \rightarrow b'(\mathbf{x}, t) \frac{\partial F}{\partial \xi} , \quad \omega = -\frac{\partial \phi'}{\partial t} \frac{1}{N^2} \frac{\partial F}{\partial \xi}$$

The mass conservation equation gives

$$\frac{\partial \phi'}{\partial t} \left[ -\frac{1}{\bar{\rho}} \frac{\partial}{\partial \xi} \frac{\bar{\rho}}{N^2} \frac{\partial F}{\partial \xi} \right] + \nabla \cdot \mathbf{u} F = 0$$
giving again the vertical structure eigenvalue equation

\[
\frac{1}{\rho} \frac{\partial}{\partial \xi} \frac{\rho}{N^2} \frac{\partial}{\partial \xi} F = -\frac{1}{gH_e} F
\]

and the horizontal equations

\[
\frac{\partial}{\partial t} u + f \mathbf{k} \times u = -\nabla \phi'
\]

\[
\frac{1}{gH_e} \frac{\partial \phi'}{\partial t} + \nabla \cdot u = 0
\]

The lower boundary condition gives the surface pressure

\[
\frac{\partial \phi}{\partial \xi} s + \phi'(x,0,t) \simeq 0 \quad \Rightarrow \quad \xi_s = \frac{1}{g} \phi'(x,t) F(0)
\]

and its evolution

\[
\frac{\partial \xi_s}{\partial t} = \omega(x,0,t) = -\frac{\partial \phi'}{\partial t} \frac{1}{N^2} \frac{\partial F}{\partial \xi} \quad \Rightarrow \quad \frac{\partial F}{\partial \xi} = \frac{N^2}{g} F \quad \text{at} \quad \xi = 0
\]

Often, however, the simpler condition \( \omega = 0 \) \( \Rightarrow \frac{\partial F}{\partial \xi} = 0 \) is used.

*Isothermal atmosphere*

One case that can be worked out completely is the isothermal basic state. Using the gas law gives \( \bar{\rho} = \rho RT \); the hydrostatic equation then gives

\[
\bar{\rho} = p_0 \exp(-z/H_s) \quad , \quad \bar{\rho} = \rho_0 \exp(-z/H_s) \quad , \quad H_s = RT/g \quad , \quad p_0 = \rho_0 g H_s
\]

—— the density decays exponentially with a scale height \( H_s \). We can just choose \( \xi = H_s \ln(p_0/p) \) so that it’s the same as height. The associated density \( \rho_c = \bar{\rho} \) as before. When we calculate the Brunt-Väisälä frequency, we get

\[
\frac{N^2}{H_s} = \frac{g}{H_s} - \frac{g^2}{c_s^2} = \frac{g}{H_s} \left[ 1 - \frac{c_v}{c_p} \right] = \frac{g}{H_s} \frac{R}{c_p}
\]

and discover that it is constant. The vertical structure equation becomes

\[
\frac{\partial^2 F}{\partial \xi^2} - \frac{1}{H_s} \frac{\partial F}{\partial \xi} = -\frac{1}{H_s H_e c_p} R F
\]

Therefore \( F \) will have exponential solutions

\[
F = \exp(\alpha z/H_s) \quad , \quad \alpha^2 - \alpha + \frac{H_s}{H_e c_p} \frac{R}{c_p} = 0 \quad , \quad \alpha = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4 \frac{R}{c_p} \frac{H_s}{H_e}}
\]
If we start with the case when the argument of the square root is positive, we must eliminate the large root, since it has an energy density $\mathcal{P}u^2 \sim \exp([2\alpha - 1]\xi/H_s)$ which grows towards infinity. Therefore we can only accept the negative sign, giving

$$
F = \exp \left( \left[ 1 - \sqrt{1 - 4 \frac{R H_s}{c_p H_e}} \frac{\xi}{2H_s} \right] \right)
$$

The lower boundary condition gives (for $\omega = 0$)

$$
\alpha = 0 \Rightarrow \frac{1}{gH_e} \to 0 \quad , \quad F = 1
$$

or for the full condition

$$
\alpha = \frac{H_s N^2}{g} = \frac{R}{c_p} \Rightarrow H_e = \frac{H_s}{1 - \frac{R}{c_p}} = \frac{c_p}{c_v} H_s \quad , \quad F = \exp\left( \frac{R}{c_p H_s} \xi \right)
$$

which will be well-behaved as long as $c_p > 2R$ (for the atmosphere $c_v$, $c_p$, $R = 718$, 1005, 287.1 J/kg/K (Tsonis, An Introduction to Atmospheric Thermodynamics) so that this condition is fine. The equivalent depth is 40% larger than the scale height. Over one scale height, $F$ grows by a factor of $\exp(R/c_p) = 1.33$ while the kinetic energy density decreases by $\exp(2\frac{R}{c_p} - 1) = 0.65$. This is called the equivalent barotropic mode.

Are there any other modes? The derivation above makes it clear that this is the only mode with $H_e > 4(R/c_p)H_s = 1.14H_s$. What about the modes with complex $\alpha$ which have energies remaining order one at infinity? The lower boundary condition clearly requires both the $\alpha_+$ and $\alpha_-$ modes; however, the latter will have downward energy flux. To maintain such a mode, we require a reflecting surface or an energy source high in the atmosphere. This will not happen for a resting atmosphere; therefore, the only mode available is the equivalent barotropic mode.
Summarizing

In the atmosphere, the standard choice is pressure coordinates

\[
\frac{D}{Dt} \mathbf{u} + f \hat{k} \times \mathbf{u} = -\nabla \phi + G(\xi)\theta \hat{k}
\]

\[
\nabla \cdot \mathbf{u} + \omega_p = 0
\]

\[
\frac{D}{Dt} \theta = 0
\]

\[
q = -g(\nabla_3 \times \mathbf{u} + f \hat{k}) \cdot \nabla_3 \ln \theta
\]

\[
Q = \nabla^2 \psi + \frac{\partial}{\partial p} \frac{f^2}{\overline{S}} \frac{\partial}{\partial p} \psi + f
\]

\[
G = -\frac{R}{p_0} \left( \frac{p}{p_0} \right)^{-1/\gamma}, \quad \overline{S} = -\frac{1}{\overline{p}} \frac{\partial}{\partial p} \ln \overline{\theta}
\]

but the log form is also convenient if we work with an isothermal stratification so that

\[
\overline{p} = \rho_c = \rho_0 \exp(-\xi/H)
\]

\[
\frac{D}{Dt} \mathbf{u} + f \hat{k} \times \mathbf{u} = -\nabla \phi + \frac{\theta}{\overline{\theta}} \hat{k}
\]

\[
\nabla \cdot \mathbf{u} + \left( \frac{\partial}{\partial \xi} - \frac{1}{H} \right) \omega = 0
\]

\[
\frac{D}{Dt} \theta = 0
\]

\[
q = \frac{1}{\overline{p}} (\nabla_3 \times \mathbf{u} + f \hat{k}) \cdot \nabla_3 \ln \theta
\]

\[
Q = \nabla^2 \psi + \frac{f^2}{N^2} \left( \frac{\partial}{\partial \xi} - \frac{1}{H} \right) \frac{\partial}{\partial \xi} \psi + f
\]

\[
\overline{p} = \rho_0 e^{-\xi/H} \quad , \quad \overline{N^2} = \frac{g \gamma - 1}{H \gamma} \quad , \quad \overline{b} = \theta_0 e^{(\gamma - 1)\xi/\gamma H}
\]

For both systems, the choice with \( \xi \) corresponding to the resting atmosphere so that \( \rho_c = \overline{p} \) gives a quasi-Boussinesq model

\[
\frac{D}{Dt} \mathbf{u} + f \hat{k} \times \mathbf{u} = -\nabla \phi + b \hat{k}
\]

\[
\nabla \cdot \mathbf{u} + \frac{1}{\overline{p}} \frac{\partial}{\partial \xi} (\overline{\rho} \omega) = 0
\]

\[
\frac{\partial}{\partial t} b + \mathbf{u} \cdot \nabla b + \omega \overline{S} = 0
\]

\[
q = \frac{1}{\overline{p}} (\nabla_3 \times \mathbf{u} + f \hat{k}) \cdot \nabla_3 \eta
\]

\[
Q = \nabla^2 \psi + \frac{1}{\overline{p}} \frac{\partial}{\partial \xi} \overline{p} \frac{f^2}{N^2} \frac{\partial}{\partial \xi} \psi + f
\]

\[
\overline{S} = \overline{N^2} = -\frac{\overline{b} \xi}{\overline{p}} - \frac{g^2}{c_s^2} \quad , \quad b = \frac{\theta}{\overline{\theta}} \quad , \quad \overline{N^2} = \frac{g}{\partial \xi} \ln \overline{\theta}
\]
For the ocean, we usually use \((p_0 - p)/\rho_0 g\) and ignore the difference between \(S\) and \(N^2\), giving a Boussinesq form

\[
\frac{D}{Dt} \mathbf{u} + f \hat{k} \times \mathbf{u} = -\nabla \varphi + \vec{b} \hat{k}
\]
\[
\nabla \cdot \mathbf{u} + \omega_\xi = 0
\]
\[
\frac{D}{Dt} \vec{b} = 0
\]
\[
q = \frac{1}{\rho_0} (\nabla_3 \times \mathbf{u} + f \hat{k}) \cdot \nabla_3 \! b
\]
\[
Q = \nabla^2 \psi + \frac{\partial}{\partial \xi} \frac{f^2}{N^2} \frac{\partial}{\partial \xi} \psi + f
\]
\[
\bar{b} = b - \frac{g^2}{c_s^2} \xi, \quad S \simeq b_\xi - \frac{g^2}{c_s^2}
\]

**Summary table**

<table>
<thead>
<tr>
<th>(\xi)</th>
<th>(\rho_c)</th>
<th>(G)</th>
<th>(\bar{S}_{\text{(atm.)}})</th>
<th>(\bar{S}_{\text{(oc.)}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>(-1/g)</td>
<td>(-\frac{R}{p_0} \left( \frac{\xi}{p_0} \right)^{-1/\gamma})</td>
<td>(-\frac{1}{p} \frac{\partial}{\partial \xi} \ln \theta)</td>
<td></td>
</tr>
<tr>
<td>((p_0 - p)/\rho_0 g)</td>
<td>(\rho_0)</td>
<td>(\frac{g}{\rho_0} \left( 1 - \frac{\xi}{H} \right)^{-1/\gamma})</td>
<td>(g \frac{\rho_0}{p} \frac{\partial}{\partial \xi} \ln \theta)</td>
<td>(\frac{\rho_0^2}{p} N^2 \simeq 7)</td>
</tr>
<tr>
<td>(-H \ln \frac{p}{p_0})</td>
<td>(\rho_0 e^{-\xi/H})</td>
<td>(\frac{g}{\rho_0} \exp \left( -\frac{\gamma - 1}{\gamma} \frac{\xi}{H} \right))</td>
<td>(g \rho_0 e^{-\xi/H} \frac{\partial}{\partial \xi} \ln \theta)</td>
<td></td>
</tr>
<tr>
<td>(\frac{H}{\gamma - 1} \left[ 1 - \left( \frac{p}{p_0} \right)^{(\gamma - 1)/\gamma} \right])</td>
<td>(\rho_0 \left[ 1 - \frac{\xi}{H} \frac{\gamma - 1}{\gamma} \right]^{1/(\gamma - 1)})</td>
<td>(\frac{g}{\rho_0})</td>
<td>(g \frac{\rho_0}{p} \left[ 1 - \frac{\xi}{H} \frac{\gamma - 1}{\gamma} \right] \frac{1}{(\gamma - 1)} \frac{\partial}{\partial \xi} \ln \theta)</td>
<td></td>
</tr>
<tr>
<td>(- \int_{p_0}^p dp' \frac{p'}{\rho(p') g} )</td>
<td>(\bar{p})</td>
<td>(\frac{g}{\theta})</td>
<td>(g \frac{\partial}{\partial \xi} \ln \theta)</td>
<td>(\frac{N^2}{\gamma - 1})</td>
</tr>
</tbody>
</table>

In this chart, \(p_0\) and \(\rho_0\) are reference values; the scale height is related to these two by \(gH = RT_0 = R\theta_0 = p_0/\rho_0\).