Orthogonal, Curvilinear Coordinates

Definition

We define a two sets of coordinates for each spatial point: $(x,\ y,\ z)$, the normal Cartesion form, and $(\xi_1,\ \xi_2,\ \xi_3)$, a second reference system. The functions ξ_i are smooth functions of the x_i coordinates, $\xi_i = \xi_i(\mathbf{x})$. We define scale factors— the length corresponding to a small displacement $d\xi_i$ by

$$h_i(\mathbf{x}) = 1/|\nabla\xi_i|$$

using the ordinary Cartesian form of the gradient. Note that the scale factors depend on position. We next define unit vectors corresponding to the displacements in each of the new coordinate directions by

$$\mathbf{\hat{e}_i'} = h_i \nabla\xi_i \tag{1}$$

We will use the following summation convention: indices which appear at least twice on one side of an equation and do not appear on the other are summed over. Thus the i index is not summed in the previous expression.

The new coordinate system is orthogonal and right–handed if

$$\mathbf{\hat{e}_i'} \cdot \mathbf{\hat{e}_j'} = \delta_{ij} \tag{2}$$

and

$$\mathbf{\hat{e}_i'} \cdot \mathbf{\hat{e}_j'} \times \mathbf{\hat{e}_k'} = \epsilon_{ijk} \tag{3}$$

Transformation of a Vector

If we have a vector given by its three components in Cartesian coordinates

$$\mathbf{F} = F_i \mathbf{\hat{e}_i}$$

we can write the components in the new coordinates

$$\mathbf{F} = F_i' \mathbf{\hat{e}_i'}$$

using the projection formula

$$F_i' = \mathbf{\hat{e}_i'} \cdot \mathbf{F} = \mathbf{\hat{e}_i'} \cdot \mathbf{\hat{e}_j} F_j$$

or

$$F_i' = \gamma_{ij} F_j \tag{4}$$

with

$$\gamma_{ij} = \mathbf{\hat{e}_i'} \cdot \mathbf{\hat{e}_j} = h_i \frac{\partial \xi_i}{\partial x_j}$$

so that

$$\mathbf{\hat{e}_i'} = \gamma_{ij} \mathbf{\hat{e}_j}$$

We can also transform backwards:

$$F_i = F_j' \gamma_{ji}$$

This follows from the orthogonality condition

$$\begin{aligned}\delta_{ij} &= \mathbf{\hat{e}'_i} \cdot \mathbf{\hat{e}'_j} \\ &= \gamma_{ik}\mathbf{\hat{e}_k} \cdot \gamma_{jm}\mathbf{\hat{e}_m} \\ &= \gamma_{ik}\gamma_{jm}\delta_{km} \\ &= \gamma_{ik}\gamma_{jk}\end{aligned} \tag{5}$$

We also have a triple product rule

$$\gamma_{im}\gamma_{jn}\gamma_{kl}\epsilon_{mnl} = \epsilon_{ijk} \tag{6}$$

The backwards transformation also implies that

$$\gamma_{ij} = \frac{1}{h_i}\frac{\partial x_j}{\partial \xi_i}$$

and

$$\gamma_{ki}\gamma_{kj} = \delta_{ij} \tag{5a}$$

Gradient

We first show that the gradient transforms as a vector:

$$\frac{\partial}{\partial x_i}\phi = \frac{\partial \xi_j}{\partial x_i}\frac{\partial}{\partial \xi_j}\phi$$

by chain rule. Multiplying and dividing by h_j gives

$$\begin{aligned}(\nabla\phi)_i &= h_j\frac{\partial \xi_j}{\partial x_i}\frac{1}{h_j}\frac{\partial}{\partial \xi_j}\phi \\ &= \gamma_{ji}(\nabla'\phi)_j\end{aligned} \tag{7}$$

where the ∇' indicates the gradient in the new coordinate system with components

$$(grad\ \phi)_j = \frac{1}{h_j}\frac{\partial}{\partial \xi_j}\phi \tag{8}$$

Multiplying (7) by γ_{ki} and using (5a), we find

$$(\nabla'\phi)_k = \gamma_{ki}(\nabla\phi)_i$$

as in (4).

Divergence

Next we consider the transformation of the divergence,

$$
\begin{aligned}
\frac{\partial}{\partial x_i} F_i &= \frac{1}{h_j} h_j \frac{\partial \xi_j}{\partial x_i} \frac{\partial}{\partial \xi_j} \gamma_{mi} F'_m \\
&= \frac{1}{h_j} \gamma_{ji} \gamma_{mi} \frac{\partial}{\partial \xi_j} F'_m + \frac{F'_m}{h_j} \gamma_{ji} \frac{\partial}{\partial \xi_j} \gamma_{mi} \\
&= \frac{1}{h_j} \frac{\partial}{\partial \xi_j} F'_j + \frac{F'_m}{h_j} \gamma_{ji} \frac{\partial}{\partial \xi_j} \frac{1}{h_m} \frac{\partial x_i}{\partial \xi_m}
\end{aligned}
$$

The second term expands to

$$
\frac{F'_m}{h_j} \gamma_{ji} \frac{\partial x_i}{\partial \xi_m} \frac{\partial}{\partial \xi_j} \frac{1}{h_m} + \frac{F'_m}{h_j^2} \frac{\partial x_i}{\partial \xi_j} \frac{1}{h_m} \frac{\partial}{\partial \xi_j} \frac{\partial x_i}{\partial \xi_m}
$$

$$
= F'_j \frac{\partial}{\partial \xi_j} \frac{1}{h_j} + \frac{F'_m}{2 h_m h_j^2} \frac{\partial}{\partial \xi_m} \frac{\partial x_i}{\partial \xi_j} \frac{\partial x_i}{\partial \xi_j}
$$

We then use the fact that

$$
\begin{aligned}
\sum_i \left(\frac{\partial x_i}{\partial \xi_j} \right)^2 &= \sum_i h_j \gamma_{ji} h_j \gamma_{ji} \\
&= h_j^2
\end{aligned}
$$

to simplify the last term to

$$
\frac{F'_m}{2 h_m h_j^2} \frac{\partial}{\partial \xi_m} h_j^2
$$

$$
\begin{aligned}
&= F'_m \frac{1}{h_m} \frac{\partial}{\partial \xi_m} \sum_j \frac{1}{2} \ln(h_j^2) \\
&= F'_m \frac{1}{h_m h_1 h_2 h_3} \frac{\partial}{\partial \xi_m} (h_1 h_2 h_3)
\end{aligned}
$$

Thus we find

$$
\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{1}{h_j} \frac{\partial}{\partial \xi_j} F'_j + F'_j \frac{\partial}{\partial \xi_j} \frac{1}{h_j} + F'_j \frac{1}{h_1 h_2 h_3} \frac{1}{h_j} \frac{\partial}{\partial \xi_j} h_1 h_2 h_3 \\
div\ \mathbf{F} &= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial \xi_j} \left(\frac{h_1 h_2 h_3}{h_j} F'_j \right) \qquad (9)
\end{aligned}
$$

Curl

Next we consider the curl and show that it transforms as a proper vector; i.e., if

$$C_m \equiv \epsilon_{mjk} \frac{\partial}{\partial x_j} F_k$$

then

$$\gamma_{im} \epsilon_{mjk} \frac{\partial}{\partial x_j} F_k$$

should be the i^{th} component of the curl,

$$(curl\ \mathbf{F})_i = C_i' \equiv \epsilon_{ijk} \frac{1}{h_j h_k} \frac{\partial}{\partial \xi_j} (h_k F_k') \tag{10}$$

We substitute

$$F_k' = \gamma_{km} F_m = \frac{1}{h_k} \frac{\partial x_m}{\partial \xi_k} F_m$$

into the last expression to find

$$C_i' = \epsilon_{ijk} \frac{1}{h_j h_k} \left(\frac{\partial x_m}{\partial \xi_k} \frac{\partial}{\partial \xi_j} F_m + F_m \frac{\partial^2 x_m}{\partial \xi_j \partial \xi_k} \right)$$

The last term gives zero contribution because it is symmetric in i and j while the ϵ_{ijk} is antisymmetric. Therefore

$$\begin{aligned} C_i' &= \epsilon_{ijk} \frac{1}{h_j h_k} \frac{\partial x_m}{\partial \xi_k} \frac{\partial}{\partial \xi_j} F_m \\ &= \epsilon_{ijk} \frac{1}{h_j h_k} \frac{\partial x_m}{\partial \xi_k} \frac{\partial x_n}{\partial \xi_j} \frac{\partial}{\partial x_n} F_m \\ &= \epsilon_{ijk} \gamma_{km} \gamma_{jn} \frac{\partial}{\partial x_n} F_m \end{aligned}$$

We thus are asking whether

$$\epsilon_{ijk} \gamma_{km} \gamma_{jn} \frac{\partial}{\partial x_n} F_m = \gamma_{im} \epsilon_{mjk} \frac{\partial}{\partial x_j} F_k$$

If we multiply both sides of this by γ_{is} and sum, we have

$$\epsilon_{ijk} \gamma_{is} \gamma_{jn} \gamma_{km} \frac{\partial}{\partial x_n} F_m = \gamma_{is} \gamma_{im} \epsilon_{mjk} \frac{\partial}{\partial x_j} F_k$$

Using (6) and (5), we find that both sides are equal to

$$\epsilon_{snm} \frac{\partial}{\partial x_n} F_m$$

So that the curl (10) indeed transforms properly.

Advective terms

Here we comment that the proper form of the advective terms is found by regarding

$$u_j \frac{\partial}{\partial x_j} u_i$$

as a shorthand for

$$grad\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2}\right) - \mathbf{u}\times curl\ \mathbf{u}$$

with the gradient and curl operations defined by (8) and (10) respectively. We cannot regard the ∇ as an ordinary vector; i.e.

$$(\nabla')_i \neq \frac{1}{h_i}\frac{\partial}{\partial \xi_i}$$

The forms (8), (9), and (11) of the gradient, divergence, and curl are clearly inconsistent with such a definition. Thus the operator

$$\mathbf{u}\cdot\nabla$$

does not, in itself, make sense. Rather we must define its forms specifically depending on the operand:

$$\mathbf{u}\cdot\nabla\phi = \mathbf{u}\cdot(grad\ \phi)$$

and

$$\mathbf{u}\cdot\nabla\mathbf{u} = grad\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2}\right) - \mathbf{u}\times curl\ \mathbf{u}$$

For advection of a different vector field, we have

$$\mathbf{u}\cdot\nabla\mathbf{B} = grad\left(\frac{\mathbf{u}\cdot\mathbf{B}}{2}\right) + \frac{1}{2}\mathbf{u}\ div\ \mathbf{B} - \frac{1}{2}\mathbf{B}\ div\ \mathbf{u} - \frac{1}{2}\mathbf{u}\times curl\ \mathbf{B} - \frac{1}{2}\mathbf{B}\times curl\ \mathbf{u} - \frac{1}{2}curl\ (\mathbf{u}\times\mathbf{B})$$

(see Morse and Feshbach).

In terms of the scale factors, we find

$$\mathbf{u}\cdot(grad\ \phi) = u_i \frac{1}{h_i}\frac{\partial}{\partial \xi_i}\phi$$

ant the i^{th} component of

$$\mathbf{u}\cdot\nabla\mathbf{u} = \frac{u_m}{h_i h_m}\frac{\partial}{\partial \xi_m} h_i u_i - \frac{u_m u_m}{h_i h_m}\frac{\partial}{\partial \xi_i} h_m = \frac{u_m}{h_m}\frac{\partial}{\partial \xi_m} u_i + \frac{u_m u_i}{h_i h_m}\frac{\partial}{\partial \xi_m} h_i - \frac{u_m u_m}{h_i h_m}\frac{\partial}{\partial \xi_i} h_m$$

as part of which we can see a term which looks like the dot product of **u** with the gradient operator and two terms which depend on the curvature of the coordinate system.