Div, Grad, Curl

Gradient

The components of the gradient are found by using the fact that the change in a scalar quantity for a small displacement is given by the dot product of the displacement vector and the gradient.

$$d\phi = d\mathbf{s} \cdot grad(\phi)$$

If we write the displacement in terms of the changes in coordinate values times the scale factors

$$d\mathbf{s} = h_i d\xi_i \hat{\mathbf{e}}'_i$$

we find

$$\begin{aligned} d\phi &= h_i d\xi_i \left[\hat{\mathbf{e}}'_i \cdot grad(\phi)\right] \\ &= h_i d\xi_i (grad\ \phi)'_i \end{aligned}$$

So that the i^{th} component of the gradient is

$$(grad\ \phi)'_i = \frac{1}{h_i} \frac{\partial}{\partial \xi_i} \phi$$

Divergence

We consider a small volume centered at point $(\xi_1,\ \xi_2,\ \xi_3)$ with sides $(d\xi_1,\ d\xi_2,\ d\xi_3)$. The divergence theorem (Gauss' theorem) gives

$$\iiint h_1 d\xi_1 h_2 d\xi_2 h_3 d\xi_3\ div\ \mathbf{F} =$$

$$\iint (h_2 h_3 F'_1)|_{\xi_1 + d\xi_1/2} d\xi_2 d\xi_3 - \iint (h_2 h_3 F'_1)|_{\xi_1 - d\xi_1/2} d\xi_2 d\xi_3 + \ldots$$

For a small volume, we can Taylor expand the right hand side and drop the integrals giving

$$h_1 d\xi_1 h_2 d\xi_2 h_3 d\xi_3\ div\ \mathbf{F} = \frac{\partial}{\partial \xi_1}(h_2 h_3 F'_1) d\xi_1 d\xi_2 d\xi_3 + \ldots$$

so that

$$div\ \mathbf{F} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial \xi_1}(h_2 h_3 F'_1) + \ldots$$

We can combine all three terms to give

$$div\ \mathbf{F} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial \xi_i}\left(\frac{h_1 h_2 h_3}{h_i} F'_i\right)$$

Curl

The curl is found by using Stokes' theorem. If we consider a small area centered at $(\xi_1,\ \xi_2,\ \xi_3)$ with sides $d\xi_1,\ d\xi_2$ and a normal $\hat{\mathbf{e}}_3'$ we have

$$(curl\ \mathbf{F})_3' h_1 d\xi_1 h_2 d\xi_2 =$$

$$(F_1 h_1)|_{\xi_2 - d\xi_2/2} d\xi_1 + (F_2 h_2)|_{\xi_1 + d\xi_1/2} d\xi_2 - (F_1 h_1)|_{\xi_2 + d\xi_2/2} d\xi_1 - (F_2 h_2)|_{\xi_1 - d\xi_1/2} d\xi_2$$

Taylor expanding gives

$$(curl\ \mathbf{F})_3' h_1 d\xi_1 h_2 d\xi_2 = -\frac{\partial}{\partial \xi_2}(F_1 h_1) d\xi_1 d\xi_2 + \frac{\partial}{\partial \xi_1}(F_2 h_2) d\xi_1 d\xi_2$$

or

$$(curl\ \mathbf{F})_3' = \frac{1}{h_1 h_2}\left[-\frac{\partial}{\partial \xi_2}(F_1 h_1) + \frac{\partial}{\partial \xi_1}(F_2 h_2)\right]$$

In general terms, then

$$(curl\ \mathbf{F})_i = \epsilon_{ijk} \frac{1}{h_j h_k} \frac{\partial}{\partial \xi_j}(h_k F_k')$$