Example of Curvilinear Coordinates – Earth coords.

We consider the case of earth coordinates: longitude (λ), latitude (θ) , and height (z). We can define these by

$$\begin{aligned} X &= (a+z)\cos\theta\sin\lambda \\ Y &= (a+z)\cos\theta\cos\lambda \\ Z &= (a+z)\sin\theta \end{aligned}$$

where a is the radius of the planet. (One could work with ellipsoids to represent the flattened shape, but it is rarely worth the effort.) The scale factors are

$$\begin{aligned} h_1 &= (a+z)\cos\theta \\ h_2 &= (a+z) \\ h_3 &= 1 \end{aligned}$$

From these relationships, we find the gradient

$$(grad\ \phi)_i = \frac{1}{h_i}\frac{\partial}{\partial \xi_i}\phi = \begin{pmatrix} \dfrac{1}{(a+z)\cos\theta}\dfrac{\partial}{\partial\lambda}\phi \\ \dfrac{1}{(a+z)}\dfrac{\partial}{\partial\theta}\phi \\ \dfrac{\partial}{\partial z}\phi \end{pmatrix}$$

The divergence can be written as

$$\begin{aligned} (div\ \mathbf{F})_i &= \frac{1}{h_1h_2h_3}\frac{\partial}{\partial\xi_j}\left(\frac{h_1h_2h_3}{h_j}F_j\right) \\ &= \frac{1}{(a+z)\cos\theta}\frac{\partial}{\partial\lambda}F_\lambda + \frac{1}{(a+z)\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta F_\theta) + \frac{1}{(a+z)^2}\frac{\partial}{\partial z}([a+z]^2F_z) \end{aligned}$$

The curl can be found from

$$(curl\ \mathbf{F})_i = \epsilon_{ijk}\frac{1}{h_jh_k}\frac{\partial}{\partial\xi_j}(h_kF_k) = \epsilon_{ijk}\frac{h_i}{h_1h_2h_3}\frac{\partial}{\partial\xi_j}(h_kF_k)$$

$$= \begin{pmatrix} \dfrac{1}{(a+z)}\dfrac{\partial}{\partial\theta}F_z - \dfrac{1}{(a+z)}\dfrac{\partial}{\partial z}[(a+z)F_\theta] \\ \dfrac{1}{(a+z)}\dfrac{\partial}{\partial z}[(a+z)F_\lambda] - \dfrac{1}{(a+z)\cos\theta}\dfrac{\partial}{\partial\lambda}F_z \\ \dfrac{1}{(a+z)\cos\theta}\dfrac{\partial}{\partial\lambda}F_\theta - \dfrac{1}{(a+z)\cos\theta}\dfrac{\partial}{\partial\theta}[\cos\theta F_\lambda] \end{pmatrix}$$

Euler eqns

From

$$\frac{\partial}{\partial t}\mathbf{u} + (\boldsymbol{\zeta} + 2\Omega) \times \mathbf{u} + \frac{1}{2}\nabla|\mathbf{u}|^2 = -\frac{1}{\rho}\nabla p - \nabla\Phi$$

we can write out the three momentum equations. The absolute vorticity is

$$\boldsymbol{\zeta}+2\Omega = \left(\frac{1}{r}w_\theta - \frac{1}{r}(rv)_z \quad , \quad \frac{1}{r}(ru)_z - \frac{1}{r\cos\theta}w_\lambda + 2\Omega\cos\theta \quad , \quad \frac{1}{r\cos\theta}v_\lambda - \frac{1}{r\cos\theta}(\cos\theta u)_\theta + 2\Omega\sin\theta\right)$$

$(r = a + z)$ and find

$$\begin{aligned}
\frac{D}{Dt}u - 2\Omega\sin\theta v + \frac{uw - uv\tan\theta}{r} + 2\Omega\cos\theta w &= -\frac{1}{\rho r\cos\theta}\frac{\partial}{\partial\lambda}p \\
\frac{D}{Dt}v + 2\Omega\sin\theta u + \frac{wv + u^2\tan\theta}{r} &= -\frac{1}{\rho r}\frac{\partial}{\partial\theta}p \\
\frac{D}{Dt}w - 2\Omega\cos\theta u - \frac{u^2 + v^2}{r} &= -\frac{1}{\rho}\frac{\partial}{\partial z}p - g
\end{aligned}$$

with $\frac{D}{Dt}$ the scalar form of the operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r\cos\theta}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial z}$$

The mass equation

$$\frac{\partial}{\partial t}\rho + \frac{1}{r\cos\theta}\frac{\partial}{\partial\lambda}(\rho u) + \frac{1}{r\cos\theta}\frac{\partial}{\partial\theta}(\rho\cos\theta v) + \frac{1}{r^2}\frac{\partial}{\partial z}(\rho r^2 w) = 0$$

the thermodynamic equation

$$\frac{D}{Dt}\rho - \frac{1}{c_s^2}\frac{D}{Dt}p = 0$$

and the equation of state

$$c_s^2 = c_s^2(\rho, p)$$

complete the system. Yes, we usually think of thermodynamics as giving an equation for T and the equation of state as $\rho(T,p)$ (adding salinity in the ocean); slightly more sophisticated versions express the density in terms of potential temperature Θ instead, because that is conserved in adiabatic motion. But you can always invert the relationships (though the 4° freshwater maximum could be an issue; see comment below however).

For an ideal gas, the speed of sound is $c_s^2 = \gamma p/\rho$ $(\gamma = c_p/c_v)$, while for the ocean it is large, so that the only significant contribution from the second term in the thermodynamic equation is from the hydrostatic pressure associated with the stratification. It brings in the $-g^2/c_s^2$ term in the Brunt-Vaisala frequency.

Laplacians

The scalar Laplacian is

$$\nabla^2\phi = \frac{1}{r^2\cos^2\theta}\frac{\partial^2}{\partial\lambda^2}\phi + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi - \frac{\sin\theta}{r^2\cos\theta}\frac{\partial}{\partial\theta}\phi + \frac{\partial^2}{\partial r^2}\phi + \frac{2}{r}\frac{\partial}{\partial r}\phi$$

The vector Laplacian acting on an eastward velocity is

$$\nabla^2 u\hat{\mathbf{e}}_\lambda = \begin{pmatrix} \frac{1}{r^2\cos^2\theta}\frac{\partial^2}{\partial\lambda^2}u + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}u - \frac{\sin\theta}{r^2\cos\theta}\frac{\partial}{\partial\theta}u - \frac{u\sin^2\theta}{r^2\cos^2\theta} + \frac{\partial^2}{\partial r^2}u - \frac{u}{r^2} + \frac{2}{r}\frac{\partial}{\partial r}u \\ \frac{2\sin\theta}{r^2\cos^2\theta}\frac{\partial}{\partial\lambda}u \\ -\frac{2}{r^2\cos\theta}\frac{\partial}{\partial\lambda}u \end{pmatrix}$$

and the difference in the eastward component becomes

$$\hat{\mathbf{e}}_\lambda \cdot \nabla^2(u\hat{\mathbf{e}}_\lambda) - \nabla^2 u = -\frac{u\,\sin^2\theta}{r^2\,\cos^2\theta} - \frac{u}{r^2}$$

not to mention the vector Laplacian having terms in the other two components. Some of these go away for a nondivergent flow:

$$\nabla^2\mathbf{u} = \begin{pmatrix} \nabla^2 u + \frac{2}{r^2\cos\theta}\frac{\partial w}{\partial\lambda} - \frac{2\sin\theta}{r^2\cos\theta}\frac{\partial v}{\partial\lambda} - \frac{\sin^2\theta\,u}{r^2\,\cos^2\theta} - \frac{u}{r^2} \\ \nabla^2 v + \frac{2}{r^2}\frac{\partial w}{\partial\theta} - \frac{\sin^2\theta\,v}{r^2\,\cos^2\theta} - \frac{v}{r^2} + \frac{2\sin\theta}{r^2\cos^2\theta}\frac{\partial u}{\partial\lambda} \\ \nabla^2 w + \frac{2}{r}\frac{\partial w}{\partial r} + \frac{2w}{r^2} \end{pmatrix}$$

but we still cannot express the zonal component of the Laplacian of the flow as an operator just on u.

For $\mathbf{u} = -curl(\psi(\lambda,\theta)\hat{\mathbf{e}}_z)$, the vorticity is just $\hat{\mathbf{e}}_z\nabla^2\psi$.