Fluid dynamics Edit

In fluid dynamics, we do not deal with the individual molecules bat rather are concerned with average properties such as the mass per unit volume (the density $\rho = \sum m_i/V$) and the velocity $\mathbf{u} = \sum m_i \mathbf{u}_i / \sum m_i$. [Bold face is used for vectors.] Here, we will just use the standard approach of treating the fluid as a **field** – all properties (density, velocity, temperature, salinity, pressure) are continuous functions of space and time. Since we deal with macroscopic volumes, this is fine for fluid molecules (and even better with quantum mechanics), but it gets shakier as we move to much more dilute properties such as the biota.

The fundamental principle in continuum dynamics for fluids is that the rate of change of the amount of some "stuff" c in a volume around point $\mathbf{x}$ is given by the difference between the amount moving into and out of the volume and the sources and sinks. The amount leaving through a small patch of surface of the volume per unit time is given by the flux, defined as the amount of c passing through a unit area in a unit time. In the following, we define c as a density, meaning amount of "stuff" per unit volume. The flux can have two parts: (1) c can be carried along with a smooth flow with velocity c, and (2) c can move across the surface by random motions. Let us consider the two cases separately, though generally both will contribute.

Advective fluxes: The example here illustrates the movement of material through a surface; if you change the angle, the flux changes because fewer particles hit the surface. The volume passing through a unit area of the surface per unit time is just $\mathbf{u} \cdot \hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the normal to the surface (see fig. 1).

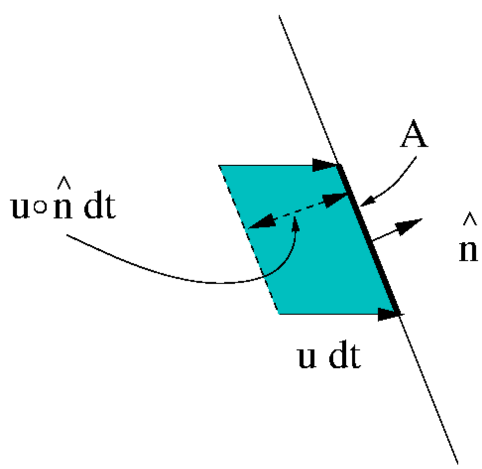

Fig. 1: Advective fluxes. The colored volume passes through area A in time dt. For density c within this volume the total amount is then $\mathbf{u} \cdot \hat{\mathbf{n}}\, cAdt$ and $|\mathbf{F}| = \mathbf{u} \cdot \hat{\mathbf{n}}\, c$.

We can thus represent the advective flux as a vectior quantity $\mathbf{F} = \mathbf{u}c$.

Diffusive fluxes: Diffusion represents an exchange of parcels across a surface with no net mass transfer (fig. 2). (Any net mass flux is incorporated in $\mathbf{u}$.) Consider a volume $A\ell$ on each side of the area A and take part of this volume δV_- from the left and swap it with δV_+ from the right. Here ℓ represents the mean free path for the random exchanges.

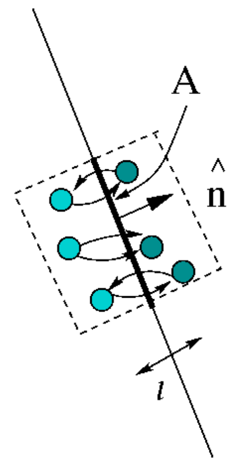

Fig. 2: Diffusive fluxes. The colored volumes pass back and forth through area A in time dt. Light-colored (low c) fluid is transferred to the right and darker (higher c) to the left resulting in a flux down the gradient.

The flux is then

$$F = -\frac{c_+\delta V_+ - c_-\delta V_-}{A\delta t}$$

Equating the masses implies $\delta M = \rho_-\delta V_- = \rho_+\delta V_+$ so that

$$F = -\frac{\delta M}{A\delta t}\left[\frac{c_+}{\rho_+} - \frac{c_-}{\rho_-}\right] = -\frac{\delta M\ell}{A\delta t}\frac{\partial}{\partial x}\frac{c}{\rho}$$

giving a vector flux

$$\mathbf{F} = -\rho\kappa\nabla\frac{c}{\rho}$$

with the diffusivity, $\kappa = \frac{\ell^2}{\delta t}\frac{\delta M}{M}$, having units of length squared over time. Here $M = \rho\ell A$ is the mass in the original volume and $\delta M/M$ represents the fraction which is exchanged across the surface in time δt. Since we are concerned with small scale exchanges, the density terms are generally ignored. For properties such as salinity (salt per unit mass), we indeed have just ∇S but the ρ terms enter in other ways.

`Examples:` advective flux , purely diffusive flux , and advection plus diffusion .

The integral of the $\mathbf{F}\cdot\hat{\mathbf{n}}$ over the surface will tell us the net amount leaving per unit time:

$$\frac{\partial}{\partial t}\int_V d\mathbf{x}\, c(\mathbf{x},t) = -\oint_S da\,\mathbf{F}\cdot\hat{\mathbf{n}} + \int_V d\mathbf{x}\,\mathcal{C}$$

Here $\mathcal{C}$ represents sources and sinks. Applying the divergence theorem gives

$$\frac{\partial}{\partial t}c = -\nabla\cdot\mathbf{F} + \mathcal{C}$$

Mass Edit

For mass, or density ρ, the flux is just $\mathbf{u}\rho$, using the fact described above that the velocities are defined by mass-weighted velocities of individual molecules.

$$\frac{\partial}{\partial t}\rho = -\nabla \cdot (\mathbf{u}\rho)$$

Salt Edit

For properties measured in a per-unit-mass form, like salinity, the conservation law becomes

$$\frac{\partial}{\partial t}\rho S = -\nabla \cdot (\mathbf{u}\rho S - \rho\kappa\nabla S)$$

Using the mass equation leads to

$$\frac{\partial}{\partial t}S = -\mathbf{u} \cdot \nabla S + \frac{1}{\rho}\nabla \cdot (\kappa\rho\nabla S)$$

This is usually written using the "material derivative"

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

(and ignoring ρ factors in the diffusion) as

$$\frac{D}{Dt}S = \nabla \cdot (\kappa\nabla S)$$

For salinity, the sources and sinks are at the sea surface (where they are really fluxes of water turning to vapor or from precipitation) or boundaries.

Momentum Edit

In Cartesian coordinates, we can look at individual components of the momentum per unit volume ρu_i

$$\frac{\partial}{\partial t}\rho u_i = -\nabla \cdot (\rho\mathbf{u}u_i) + \nabla \cdot (\nu\rho\nabla u_i) + F_i$$

or

$$\frac{\partial}{\partial t}u_i = -\mathbf{u} \cdot \nabla u_i + \nu\nabla^2 u_i + \frac{1}{\rho}F_i$$

$$\frac{D}{Dt}u_i = \nu\nabla^2 u_i + \frac{1}{\rho}F_i$$

In non-Cartesian geometries (e.g. the spherical Earth), this form is not appropriate; it does not take into account the momentum changes required for a particle to move along a coordinate line (e.g. a latitude circle). The $\frac{D}{Dt}$ operator is fine for scalars, but trickier for vectors. Therefore, we rephrase the momentum equations using the gradient, divergenve, and curl operators to give a coordinate-free description. Appendix A proves the equivalence of the form below and gives the forms of the operators in common coordinate systems. The mass and momentum equations become

$$\frac{\partial}{\partial t}\rho + div(\rho \mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}\mathbf{u} + \boldsymbol{\zeta} \times \mathbf{u} + grad(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}) = -\nu \; curl(\boldsymbol{\zeta}) + \frac{1}{\rho}\mathbf{F} \quad , \quad \boldsymbol{\zeta} = curl(\mathbf{u})$$

This form drops the subtleties of the viscous term and the way density and compressibility come into it. Here, $\boldsymbol{\zeta}$ is the vorticity and measures the local spin of the fluid. Vorticity plays a central role in geophysical fluid dynamics and, indeed, in turbulent flows

Forces Edit

The forces acting on a rotating stratified fluid are gravity (which appears as buoyancy forces), pressure, Coriolis, and viscous stresses. We need to represent each of these as the force exerted per unit volume.

Pressure represents the forces that the molecules exert as they bounce off each other during their thermal fluctuations (not the average velocity $\mathbf{u}$). Conceptually, if we consider a wall in a fluid with no average motion, each time a molecule bounces off a wall, it applies a force to the wall (and the wall applies an equal and opposite force to reverse the normal component of the molecule's velocity). The net force is the product of the average normal velocity, the mass of the molecules, and the number hitting the wall per unit time. If we double the size of the wall, we double the number of molecules impinging on it, and double the force. To account for this, we define the pressure as the force per unit area.

Now consider the forces on a small cube-shaped object centered on location x in the fluid. If the thermal motion is the same everywhere in the fluid, the forces exerted on the box by molecules bouncing off the left wall will be equal and opposite to that exerted by molecules bouncing off the right wall. Therefore the net force on the cube will be zero. But if the speeds of the molecules on the right are higher than that of those on the left, the force on the right side of the box pushing it to the left will be greater than the force on the left side pushing it to the right. The non-zero net force depends on changes in pressure and will try to push the box towards the lower pressure regions. The same argument applies if we replace the solid box with a parcel of fluid; if the molecules on the right are moving faster, collisions with them will apply more force on the fluid parcel than those with the molecules on the left. Thus we can see that the force depends on the gradient of the pressure.

To formalize this, we use the definition of pressure, as the normal force per unit area exerted by fluid outside a volume on the fluid inside, to write

$$\mathbf{F}V = \int_{\partial V} -p\hat{\mathbf{n}} d^2\mathbf{x} \quad \Rightarrow \quad F_1 = -\frac{1}{V}\int_{\partial V} p\hat{\mathbf{x}} \cdot \hat{\mathbf{n}} \, d^2\mathbf{x} = -\frac{1}{V}\int_{V} \nabla \cdot (\hat{\mathbf{x}}p) \, d^3\mathbf{x}$$

In the limit, the force per unit volume is

$$F_1 = -\frac{\partial p}{\partial x} \quad \Rightarrow \quad \mathbf{F} = -\nabla p$$

CORIOLIS "forces" act on matter moving in a rotating system. A particle moving horizontally but subject to no real horizontal forces appears to move in a curved path because the Earth is rotating under it. Consider a satellite starting over England given a push due northward in a polar orbit. It has no east-west or north-south acceleration, just gravity holding in the orbit. What does the track (marked by periodically dropped paintballs) look like relative to the ground? This animation of very slow polar orbit shows the result: the track appears curved. We ascribe this curvature to a fictitious force perpendicular to the track – the Coriolis force.

Suppose we consider three snapshots of a particle subject to no external forces viewed in both a fixed (inertial) and a rotating frame of reference. In inertial space , the particle is moving in a straight line; we set $t = 0$ as the time when it passes through the origin heading along the x-axis. In the inertial (fixed) frame, its position is given by

$$\mathbf{x}_f = (u_0 t, 0, 0)$$

giving successive points

$$\mathbf{x}_f : \ (-u_0 \delta t, 0, 0) \rightarrow (0, 0, 0) \rightarrow (u_0 \delta t, 0, 0)$$

Correspondingly, the positions in the rotating frame are

$$\mathbf{x} : \ (-u_0 \delta t \cos(\Omega \delta t), -u_0 \delta t \sin(\Omega \delta t), 0) \rightarrow (0, 0, 0) \rightarrow$$

$$(u_0 \delta t \cos(\Omega \delta t), -u_0 \delta t \sin(\Omega \delta t), 0)$$

where Ω is the rotation rate of the reference frame.

Clearly the particle accelerates in the $-y$ direction. Indeed, for this case, using an approximation to the second derivative gives

$$\begin{aligned}
\frac{d^2\mathbf{x}}{dt^2} &\simeq \frac{[\mathbf{x}(t+\delta t) - \mathbf{x}(t)] - [\mathbf{x}(t) - \mathbf{x}(t-\delta t)]}{\delta t^2} \\
&= \frac{\mathbf{x}(t+\delta t) + \mathbf{x}(t-\delta t) - 2\mathbf{x}(t)}{\delta t^2} \\
&= \frac{(0, -2u_0 \delta t \sin(\Omega \delta t), 0)}{\delta t^2} \\
&= -2\Omega u_0 \hat{\mathbf{y}} = -2\vec{\Omega} \times \mathbf{u}
\end{aligned}$$

Applying the same argument to a particle moving north shows that it also accelerates to the right.

If we were to postulate some force as causing this acceleration, the strength would be

$$\mathbf{F} = -\rho 2\vec{\Omega} \times \mathbf{u}$$

This Coriolis "force" is of course an artifact of dealing with movement in an accelerating reference frame (remember that circular motion has a velocity vector which is constantly changing with time) but it can be used just as though it were real. Usually, however, we will put this term on the left-hand side to keep it with the accelerations relative to the earth

$$\frac{D}{Dt}\mathbf{u} + 2\vec{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{F}$$

with F including gravity/centrifugal forces (see Appendix B) and viscous stresses.

GRAVITY: The effects of gravity are straight-forward: the force is minus the mass times the gradient of the geopotential giving a force per unit volume

$$\mathbf{F} = -\rho\nabla\Phi$$

By definition, this is in the vertical direction, so we usually take $\Phi = gz$; variations of g with latitude (because of centrifugal terms) are small but can be important for some problems. We will take g to be constant and write

$$\mathbf{F} = -\rho g\hat{\mathbf{z}}$$

VISCOUS STRESSES are tangential forces acting across a surface; conceptually, a faster moving (on average) eastward stream located (for example) to the north of a slower stream will impart some of its momentum to the slower stream by collisions between the molecules, in effect exerting an eastward force. The slower stream has the opposite effect on the faster one. Thus, the tendency is to equalize the velocities; the stresses act much like diffusion of velocity

$$\mathbf{F} = \rho\nu\nabla^2\mathbf{u}$$

where ν is the kinematic viscosity having units (like diffusivity) of L^2/T.

MOMENTUM EQUATIONS: Putting all the forces together gives the momentum equations

$$\frac{D}{Dt}\mathbf{u} + 2\vec{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} - g\hat{\mathbf{z}}$$

or

$$\frac{\partial}{\partial t}\mathbf{u} + (2\vec{\Omega} + \boldsymbol{\zeta}) \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla(\frac{1}{2}|\mathbf{u}|^2 + \Phi) + \nu\nabla^2\mathbf{u} \tag{1}$$

Thermodynamics, buoyancy, and the Boussinesq approximation Edit

The momentum and mass equations are not sufficient to predict the evolution of the flow: given the current state at time t, we know how $\mathbf{u}$ and ρ change with time but cannot determine p at $t + \delta t$. Fluids have an **equation of state** relating the density to other properties including the pressure; for seawater, this is expressed as

$$\rho = \rho(S, T, p)$$

where S is the salinity (grams of salt per kilogram of seawater) and T is the temperature. If ρ were only a function of pressure, we could invert the relationship to find the new pressure given the new density; however, the dependence on T and S implies we need two additional evolution equations.

For simplicity, we shall avoid these complications and make the Boussinesq approximation. We let

$$\rho \equiv \rho_0(z)(1 - \alpha\theta)$$

The quantity $\alpha g\theta = g\frac{\rho_0 - \rho}{\rho_0}$ represents the buoyant acceleration, upwards when the density is lower than average and downwards when it is higher; in the fluid, the effects of gravity are much reduced – most of it is compensated for by pressure forces. The $\rho_0(z)$ takes into account the most significant part of the compressibility of sea water, the overall increase in density with depth. If we treat salt and heat separately, then we'd use $\alpha\theta = \alpha T - \beta S$; people also use a full equation of state $\rho(T_{pot}, S, p)$ as well.

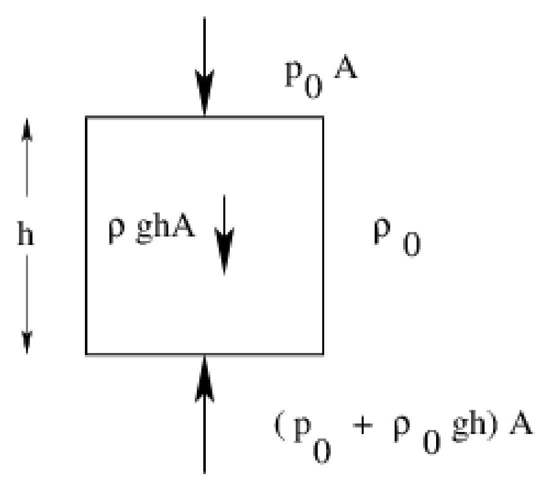

Figure 3: The net force per unit mass is $(-p_0A + p_0A + \rho_0 ghA - \rho ghA)/\rho hA = g(\rho_0 - \rho)/\rho$.

We also define a pressure-like quantity ϕ such that

$$p = -\int^z \rho_0 g + \rho_0\phi$$

so that the pressure gradient and gravitational terms become

$$
\begin{aligned}
-\frac{1}{\rho}\nabla p - g\hat{\mathbf{z}} &= -\frac{1}{1-\alpha\theta}\nabla\phi + g\left(\frac{1}{1-\alpha\theta} - 1\right)\hat{\mathbf{z}} - \frac{\phi}{1-\alpha\theta}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z}\hat{\mathbf{z}} \\
&= -\frac{1}{1-\alpha\theta}\nabla\phi + \frac{\alpha g\theta}{1-\alpha\theta}\hat{\mathbf{z}} - \frac{\phi}{1-\alpha\theta}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z}\hat{\mathbf{z}} \\
&\simeq -\nabla\phi + \alpha g\theta\,\hat{\mathbf{z}}
\end{aligned}
$$

where the last step assumes that $\alpha\theta$ and N^2H/g are small.

The thermodynamic and salinity equations give

$$\frac{D}{Dt}\theta = \kappa\nabla^2\theta + \mathcal{H}$$

where $\mathcal{H}$ represents buoyancy sources from heating or freshening. We've assumed that (1) both the flow speed and $\sqrt{gH}$ (the long surface wave speed) are small compared to the sound speed and (2) κ represents small scale mixing which transfers heat and salt similarly rather than the molecular processes which give quite different diffusivities.

Neglecting terms of similar order in the mass conservation equation shows that the flow is nearly non-divergent. Putting these equations together gives the Boussinesq system:

$$
\begin{aligned}
\frac{D}{Dt}\mathbf{u} + 2\vec{\Omega}\times\mathbf{u} &= -\nabla\phi + \alpha g\theta\,\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u} \\
\nabla\cdot\mathbf{u} &= 0 \\
\frac{D}{Dt}\theta &= \kappa\nabla^2\theta + \mathcal{H}
\end{aligned}
\qquad (Bouss)
$$

Primitive equations Edit

For many kinds of motion, the horizontal scale is much larger than the vertical. By "scale" we mean the estimated variance in a field divided by the variance of the gradient $|\frac{\partial\phi}{\partial x}| \sim \frac{1}{L}|\phi|$. We can estimate the sizes of terms in the continuity (mass) equation as

$$
\begin{array}{ccc}
\dfrac{\partial u}{\partial x} & + \dfrac{\partial w}{\partial z} & = 0 \\
\dfrac{U}{L} & \dfrac{W}{H} &
\end{array}
$$

Since we don't expect the flow to be independent of x, the vertical velocity will be order $W = UH/L$ and will be small if the horizontal scale L is much larger than the vertical scale H. The horizontal momentum equation, with time scale order L/U, has sizes like

$$
\begin{array}{ccccc}
\dfrac{\partial u}{\partial t} & + u\dfrac{\partial u}{\partial x} & + w\dfrac{\partial u}{\partial z} & = & -\dfrac{\partial\phi}{\partial x} \\
\dfrac{U^2}{L} & \dfrac{U^2}{L} & \dfrac{UH}{L}\dfrac{U}{H} & & \dfrac{\Phi}{L}
\end{array}
$$

The pressure will scale like $\Phi \sim U^2$. The vertical momentum is a different story

$$\begin{array}{ccccccc} \dfrac{\partial w}{\partial t} & + u\dfrac{\partial w}{\partial x} & + w\dfrac{\partial w}{\partial z} & = & -\dfrac{\partial \phi}{\partial z} & + \alpha g \theta \\ \dfrac{U^2 H}{L^2} & \dfrac{U^2 H}{L^2} & \dfrac{U^2 H}{L^2} & & \dfrac{U^2}{H} & ? \end{array}$$

The acceleration terms are order H^2/L^2 smaller than the pressure gradient, and the density anomalies will scale like U^2/gH which is also generally small ($\sqrt{gH}$ in the deep ocean is order 200 m/s). The vertical momentum equation becomes hydrostatic, so that the density field tells us a lot about the pressure (but not everything). The resulting equations

$$\frac{D}{Dt}u - fv = -\frac{\partial \phi}{\partial x} + \nu \nabla^2 u$$

$$\frac{D}{Dt}v + fu = -\frac{\partial \phi}{\partial y} + \nu \nabla^2 v$$

$$\frac{\partial \phi}{\partial z} = \alpha g \theta$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{D}{Dt}\theta = \mathcal{H} + \kappa \nabla^2 \theta$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

This set is the most commonly used form of the equations of motion for motions on the mesoscale and up (and is not bad for submesoscale)

Appendix A: Vorticity-Bernoulli form Edit

The Cartesian form of momentum advection is

$$u_j \frac{\partial u_i}{\partial x_j} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Later, we'll split the rate-of-strain matrix into symmetric and antisymmetric parts; for now, however, remove the antisymmetric part explicitly

$$\begin{aligned} u_j \frac{\partial u_i}{\partial x_j} &= \left[\begin{pmatrix} 0 & u_y - v_x & u_z - w_x \\ v_x - u_y & 0 & v_z - w_y \\ w_x - u_z & w_y - v_z & 0 \end{pmatrix} + \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} \right] \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ &= \left[\begin{pmatrix} 0 & -\zeta_3 & \zeta_2 \\ \zeta_3 & 0 & -\zeta_1 \\ -\zeta_2 & \zeta_1 & 0 \end{pmatrix} + \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} \right] \begin{pmatrix} u \\ v \\ w \end{pmatrix} \end{aligned}$$

The last term is just $\nabla[\frac{1}{2}|\mathbf{u}|^2$ while the first term is a rotation matrix around the vector $\boldsymbol{\zeta}$

$$\begin{pmatrix} 0 & -\zeta_3 & \zeta_2 \\ \zeta_3 & 0 & -\zeta_1 \\ -\zeta_2 & \zeta_1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\zeta_3 v + \zeta_2 w \\ \zeta_3 u - \zeta_1 w \\ -\zeta_2 u + \zeta_1 v \end{pmatrix} = \boldsymbol{\zeta} \times \mathbf{u}$$

Thus

$$u_j \frac{\partial u_i}{\partial x_j} = \boldsymbol{\zeta} \times \mathbf{u} + \nabla \frac{1}{2} |\mathbf{u}|^2$$

Earth coordinates Edit

For Earth coordinates (u=zonal, v=meridional, w=vertical), we have From these relationships, we find the gradient

$$grad\ \phi = \begin{pmatrix} \frac{1}{(a+z)\cos\theta} \frac{\partial}{\partial \lambda} \phi \\ \frac{1}{(a+z)} \frac{\partial}{\partial \theta} \phi \\ \frac{\partial}{\partial z} \phi \end{pmatrix}$$

The divergence can be written as

$$div\ \mathbf{F} = \frac{1}{(a+z)\cos\theta} \frac{\partial}{\partial \lambda} F_\lambda + \frac{1}{(a+z)\cos\theta} \frac{\partial}{\partial \theta} (\cos\theta F_\theta) + \frac{1}{(a+z)^2} \frac{\partial}{\partial z} ([a+z]^2 F_z)$$

and the curl is

$$curl\ \mathbf{F} = \begin{pmatrix} \frac{1}{(a+z)} \frac{\partial}{\partial \theta} F_z - \frac{1}{(a+z)} \frac{\partial}{\partial z} [(a+z) F_\theta] \\ \frac{1}{(a+z)} \frac{\partial}{\partial z} [(a+z) F_\lambda] - \frac{1}{(a+z)\cos\theta} \frac{\partial}{\partial \lambda} F_z \\ \frac{1}{(a+z)\cos\theta} \frac{\partial}{\partial \lambda} F_\theta - \frac{1}{(a+z)\cos\theta} \frac{\partial}{\partial \theta} [\cos\theta F_\lambda] \end{pmatrix}$$

Euler eqns Edit

From

$$\frac{\partial}{\partial t}\mathbf{u} + (\boldsymbol{\zeta} + 2\Omega) \times \mathbf{u} + \frac{1}{2}\nabla|\mathbf{u}|^2 = -\frac{1}{\rho}\nabla p - \nabla\Phi$$

we can write out the three momentum equations. The absolute vorticity is

$$\boldsymbol{\zeta} + 2\Omega = \left(\frac{1}{r}w_\theta - \frac{1}{r}(rv)_z \ , \ \frac{1}{r}(ru)_z - \frac{1}{r\cos\theta}w_\lambda + 2\Omega\cos\theta \ , \right.$$

$$\left. \frac{1}{r\cos\theta}v_\lambda - \frac{1}{r\cos\theta}(\cos\theta u)_\theta + 2\Omega\sin\theta \right)$$

$(r = a + z)$ and find

$$\begin{aligned}
\frac{D}{Dt}u - 2\Omega\sin\theta v + \frac{uw - uv\tan\theta}{r} + 2\Omega\cos\theta w &= -\frac{1}{\rho r\cos\theta}\frac{\partial}{\partial\lambda}p \\
\frac{D}{Dt}v + 2\Omega\sin\theta u + \frac{wv + u^2\tan\theta}{r} &= -\frac{1}{\rho r}\frac{\partial}{\partial\theta}p \\
\frac{D}{Dt}w - 2\Omega\cos\theta u - \frac{u^2 + v^2}{r} &= -\frac{1}{\rho}\frac{\partial}{\partial z}p - g
\end{aligned}$$

with $\frac{D}{Dt}$ the scalar form of the operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r\cos\theta}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial z}$$

The mass equation

$$\frac{\partial}{\partial t}\rho + \frac{1}{r\cos\theta}\frac{\partial}{\partial\lambda}(\rho u) + \frac{1}{r\cos\theta}\frac{\partial}{\partial\theta}(\rho\cos\theta v) + \frac{1}{r^2}\frac{\partial}{\partial z}(\rho r^2 w) = 0$$

the thermodynamic equation

$$\frac{D}{Dt}\rho - \frac{1}{c_s^2}\frac{D}{Dt}p = 0$$

and the equation of state

$$c_s^2 = c_s^2(\rho, p)$$

complete the system.

Laplacians Edit

The scalar Laplacian is

$$\nabla^2\phi = \frac{1}{r^2\cos^2\theta}\frac{\partial^2}{\partial\lambda^2}\phi + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi - \frac{\sin\theta}{r^2\cos\theta}\frac{\partial}{\partial\theta}\partial hi + \frac{\partial^2}{\partial r^2}\phi + \frac{2}{r}\frac{\partial}{\partial r}\phi$$

The vector Laplacian acting on an eastward velocity is

$$\nabla^2 u\hat{\mathbf{e}}_\lambda = \begin{pmatrix} \frac{1}{r^2\cos^2\theta}\frac{\partial^2}{\partial\lambda^2}u + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}u - \frac{\sin\theta}{r^2\cos\theta}\frac{\partial}{\partial\theta}u - \frac{u\sin^2\theta}{r^2\cos^2\theta} + \frac{\partial^2}{\partial r^2}u - \frac{u}{r^2} + \frac{2}{r}\frac{\partial}{\partial r}u \\ \frac{2\sin\theta}{r^2\cos^2\theta}\frac{\partial}{\partial\lambda}u \\ -\frac{2}{r^2\cos\theta}\frac{\partial}{\partial\lambda}u \end{pmatrix}$$

and the difference in the eastward component becomes

$$\hat{\mathbf{e}}_\lambda\cdot\nabla^2(u\hat{\mathbf{e}}_\lambda) - \nabla^2 u = -\frac{u\,\sin^2\theta}{r^2\,\cos^2\theta} - \frac{u}{r^2}$$

not to mention the vector Laplacian having terms in the other two components. Some of these go away for a nondivergent flow:

$$\nabla^2\mathbf{u} = \begin{pmatrix} \nabla^2 u + \frac{2}{r^2\cos\theta}\frac{\partial w}{\partial\lambda} - \frac{2\sin\theta}{r^2\cos\theta}\frac{\partial v}{\partial\lambda} - \frac{\sin^2\theta\,u}{r^2\,\cos^2\theta} - \frac{u}{r^2} \\ \nabla^2 v + \frac{2}{r^2}\frac{\partial w}{\partial\theta} - \frac{\sin^2\theta\,v}{r^2\,\cos^2\theta} - \frac{v}{r^2} + \frac{2\sin\theta}{r^2\cos^2\theta}\frac{\partial u}{\partial\lambda} \\ \nabla^2 w + \frac{2}{r}\frac{\partial w}{\partial r} + \frac{2w}{r^2} \end{pmatrix}$$

but we still cannot express the zonal component of the Laplacian of the flow as an operator just on u.

For $\mathbf{u} = -curl(\psi(\lambda,\theta)\hat{\mathbf{e}}_z)$, the vorticity is just $\hat{\mathbf{e}}_z\nabla^2\psi$.

Appendix B Edit

For a more formal derivation of Coriolis and centrifugal forces, let us define the coordinate rotation (aligning the z-axis along the rotation vector) as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

or

$$\mathbf{x}' = \mathbf{R}\mathbf{x}$$

where $\mathbf{x}'$ is the inertial frame coordinate and $\mathbf{x}$ is in the rotating frame. Then

$$\mathbf{u}' = \mathbf{R}\mathbf{u} + \frac{\partial\mathbf{R}}{\partial t}\mathbf{x}$$

and

$$\frac{D}{Dt}\mathbf{u}' = \mathbf{R}\frac{D}{Dt}u + 2\frac{\partial \mathbf{R}}{\partial t}\mathbf{u} + \frac{\partial^2 \mathbf{R}}{\partial t^2}\mathbf{x} = \frac{1}{\rho}\mathbf{F}' = \frac{1}{\rho}\mathbf{R}\mathbf{F}$$

Multiplying by $\mathbf{R}^{-1}$ gives

$$\frac{D}{Dt}\mathbf{u} + 2\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial t}\mathbf{u} + \mathbf{R}^{-1}\frac{\partial^2 \mathbf{R}}{\partial t^2}\mathbf{x} = \frac{1}{\rho}\mathbf{F}$$

The two matrices are

$$\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial t} = \begin{pmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathbf{R}^{-1}\frac{\partial^2 \mathbf{R}}{\partial t^2} = -\Omega^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

giving

$$\frac{D}{Dt}\mathbf{u} + 2\Omega \times \mathbf{u} = \frac{1}{\rho}\mathbf{F} - \Omega \times (\Omega \times \mathbf{x})$$

The centrifugal terms can be written as the gradient of a potential

$$-\Omega \times (\Omega \times \mathbf{x}) = grad(\Phi_c) \quad , \quad \Phi_c = \frac{1}{2}|\Omega|^2|\mathbf{x}|^2 - \frac{1}{2}(\Omega \cdot \mathbf{x})^2$$

This is combined with the gravitational potential.