Green's functions [1D]

The general second order problem

$$a(x)\frac{\partial^2}{\partial x^2}F(x) + b(x)\frac{\partial}{\partial x}F(x) + c(x)F(x) = S(x) \tag{1}$$

with (for convenience) Dirichlet boundary conditions: 0 at the left and right (which could be at infinity) has the solution

$$F(x) = \int dx' G(x,x')S(x')$$

where the Green's function satisfies

$$a(x)\frac{\partial^2}{\partial x^2}G(x,x') + b(x)\frac{\partial}{\partial x}G(x,x') + c(x)G(x,x') = \delta(x-x') \tag{2}$$

The functions a, b, and c are assumed to be smooth with $a \neq 0$.

Matching the order of singularities indicates that $\frac{\partial}{\partial x}G$ could have a step at $x = x'$, so that G itself may have a slope discontinuity but must be continuous. Integrating from $x' - \epsilon/2$ to $x' + \epsilon/2$ – across the discontinuity in $\frac{\partial}{\partial x}G$ gives

$$a(x')\left[G_x(x'+\epsilon/2,x') - G_x(x'-\epsilon/2,x')\right] + b(x')\left[G(x'+\epsilon/2,x') - G(x'-\epsilon/2,x')\right]$$

$$+\epsilon c(x')G(x',x') + HOT = 1$$

Collecting just the order one terms gives

$$a(x')[G_x(x'+\epsilon/2,x') - G_x(x'-\epsilon/2,x')] = 1 \tag{3}$$

using the continuity of G and finiteness of G_x to show that the b term is also order ϵ.

Now let's find G in terms of the free solutions to

$$a(x)\frac{\partial^2}{\partial x^2}f(x) + b(x)\frac{\partial}{\partial x}f(x) + c(x)f(x) = 0 \tag{4}$$

We can pick the two solutions $f_L(x)$ which goes to 0 at the left and $f_R(x)$ which goes to zero at the right. Then

$$G(x,x') = Af_L(x_<)f_R(x_>)$$

where $x_< = min(x,x')$ and $x_> = max(x,x')$. This satisfies the equation away from $x = x'$ since each f does; it also satisfies the boundary conditions; and it's continuous at $x = x'$ where $x_< = x_>$. The matching condition (2) is just

$$a(x')A\left[f_L(x')\frac{\partial}{\partial x}f_R(x') - f_R(x')\frac{\partial}{\partial x}f_L(x')\right] = 1$$

or

$$A = \frac{1}{a(x')W(f_L, f_R)} \tag{4}$$

with W the Wronskian defined as in the square bracket above.

We can find the Wronskian: if we use equation (4) for f_R and multiply by f_L and then subtract the product of f_R with the equation written for f_L, we have

$$a\left[f_L \frac{\partial^2}{\partial x^2} f_R - f_R \frac{\partial^2}{\partial x^2} f_L\right] + b\left[f_L \frac{\partial}{\partial x} f_R - f_R \frac{\partial}{\partial x} f_L\right] = 0$$

or, if we pull one derivative out from the first bracket (noting that the estra terms cancel),

$$a\frac{\partial}{\partial x} W + bW = 0$$

which implies

$$W(x) = C \exp\left(-\int^x dz\, \frac{b(z)}{a(z)}\right)$$

(hence requiring $a \neq 0$); we can evaluate it at a single point to find C.

For the case at hand, $a = 1$, $b = 0$ and $c = -k^2$, so that W is just constant. Since $f_L = exp(kx)$, $f_R = exp(-kx)$, we find $W = -2k$. Thus

$$G(x, x') = -\frac{1}{2k} \exp(kx_< - kx_>)$$

For $x < x'$, $x_< - x_> = x - x'$; for $x > x'$, $x_< - x_> = x' - x$; both of these are negative, so that $x_< - x_> = -|x - x'|$ and

$$G(x, x') = -\frac{1}{2k} \exp(-k|x - x'|)$$

For a more complex example, suppose we think of the polar problem $a = 1$, $b = 1/r$, $c = -m^2/r^2 - k^2$. The solutions are modified Bessel functions

$$f_L = I_m(kr) \quad , \quad f_R = K_m(kr)$$

and $W = C/r$. Using the expressions near the origin for $I_m \sim (r/2)^m/\Gamma(m+1)$ and $K_m \sim \frac{1}{2}\Gamma(m)(r/2)^{-m}$ leads to $W = -1/r$ and

$$G(r, r') = -\frac{1}{r'} I_m(r_<) K_m(r_>)$$