Laplace Tidal Equations

Basic equations

$$\frac{\partial}{\partial t}\mathbf{u} + (2\Omega + \boldsymbol{\zeta}) \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla(\phi + \frac{1}{2}\mathbf{u}\cdot\mathbf{u})$$
$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\mathbf{u}) = 0$$
$$\frac{D}{Dt}\rho - \frac{1}{c_s^2}\frac{D}{Dt}p = 0$$

The hydrostatic state corresponds to

$$\frac{1}{\overline{\rho}}\nabla\overline{p} = -\nabla\phi \quad \Rightarrow \quad \overline{p} = \overline{p}(\phi/g) \;\;,\;\; \overline{\rho} = \overline{\rho}(\phi/g) \;\; with \;\; \nabla\phi = g\hat{\mathbf{k}}$$

Expanding the right-hand side pressure and gravitational potential terms using $p \equiv \overline{p} + \overline{\rho}P'$ and $\rho = \overline{\rho} + \rho'$ gives

$$-\frac{1}{\rho}\nabla p - \nabla\phi = \frac{1}{\overline{\rho}+\rho'}\overline{\rho}\nabla\phi - \frac{1}{\overline{\rho}+\rho'}\nabla\overline{\rho}P' - \frac{\overline{\rho}+\rho'}{\overline{\rho}+\rho'}\nabla\phi = -\frac{\overline{\rho}}{\overline{\rho}+\rho'}\nabla P' - P'\frac{\nabla\overline{\rho}}{\overline{\rho}+\rho'} - \frac{\rho'}{\overline{\rho}+\rho'}\nabla\phi$$

Consistent with the linearization to come, we will now keep only first order in perturbation variables

$$-\frac{1}{\rho}\nabla p - \nabla\phi \simeq -\nabla P' - P'\frac{\nabla\overline{\rho}}{\overline{\rho}} - \frac{\rho'}{\overline{\rho}}\nabla\phi = -\nabla P' + b'\hat{\mathbf{k}}$$

where the buoyancy perturbations are given by

$$b' = -\frac{g\rho'}{\overline{\rho}} - \frac{1}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial z}P'$$

Thus we get the linearized momentum equations

$$\frac{\partial}{\partial t}\mathbf{u}' + 2\Omega \times \mathbf{u}' = -\nabla P' + b'\hat{\mathbf{k}} \tag{1}$$

The mass equation linearizes to

$$\frac{\partial}{\partial t}\frac{\rho'}{\overline{\rho}} + \frac{1}{\overline{\rho}}\nabla\cdot(\overline{\rho}\mathbf{u}') = 0$$

or

$$-\frac{\partial}{\partial t}\frac{b'}{g} - \frac{1}{g}\frac{1}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial z}\frac{\partial}{\partial t}P' + \frac{1}{\overline{\rho}}\nabla\cdot(\overline{\rho}\mathbf{u}') = 0 \tag{2}$$

Finally, the thermodynamic equation gives

$$\frac{\partial}{\partial t}(\rho' - \frac{\overline{\rho}P'}{\overline{c}_s{}^2}) + w'(\frac{\partial\overline{\rho}}{\partial z} + \frac{g\overline{\rho}}{\overline{c}_s{}^2}) = 0$$

If we multiply by $-g/\overline{\rho}$, we find

$$\frac{\partial}{\partial t}(b' - \frac{N^2}{g}P') + w'N^2 = 0 \tag{3}$$

with the Brunt-Väisälä frequency given by

$$N^2 = -\frac{g}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial z} - \frac{g^2}{\overline{c}_s{}^2}$$

Geometric approximation

For a thin shell, we can replace Ω by $f\hat{\mathbf{k}}$ and maintain energetic consistency. This may break down near the equator; elsewhere the parts of the Coriolis force associated with the local horizontal component of rotation are negligible. Therefore we shall work with (1-3) in the form

$$\frac{\partial}{\partial t}\mathbf{u}' + f\hat{\mathbf{k}} \times \mathbf{u}' = -\nabla P' + b'\hat{\mathbf{k}}$$

$$-\frac{\partial}{\partial t}\frac{b'}{g} - \frac{1}{g}\frac{1}{\overline{\rho}}\frac{\partial \overline{\rho}}{\partial z}\frac{\partial}{\partial t}P' + \frac{1}{\overline{\rho}}\nabla \cdot (\overline{\rho}\mathbf{u}') = 0$$

$$\frac{\partial}{\partial t}(b' - \frac{N^2}{g}P') + w'N^2 = 0$$

We shall also ignore the derivatives of radius in the metric terms so that

$$\frac{1}{r} \rightarrow \frac{1}{a}$$

where a is the planetary radius, and terms such as

$$\frac{1}{\overline{\rho}r^2}\frac{\partial}{\partial r}\overline{\rho}r^2 w' = \frac{1}{\overline{\rho}(a+z)^2}\frac{\partial}{\partial z}\overline{\rho}(a+z)^2 w' \rightarrow \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}w'$$

Given these approximations, we can split the horizontal and vertical parts out

$$\frac{\partial}{\partial t}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla P \qquad (f.1)$$

$$\frac{\partial}{\partial t}w = -\frac{\partial}{\partial z}P + b \qquad (f.2)$$

$$-\frac{\partial}{\partial t}\frac{b}{g} - \frac{1}{g}\frac{1}{\overline{\rho}}\frac{\partial \overline{\rho}}{\partial z}\frac{\partial}{\partial t}P + \nabla \cdot \mathbf{u} + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}(\overline{\rho}w) = 0 \qquad (f.3)$$

$$\frac{\partial}{\partial t}(b - \frac{N^2}{g}P) + wN^2 = 0 \qquad (f.4)$$

where $\mathbf{u}$ is the horizontal velocity (along geopotential surfaces) $(\mathbf{u}' - w'\hat{\mathbf{k}})$ and the gradient and divergence are likewise horizontal operators with no dependence on z. We've dropped all the primes on the wave quantities. The mass conservation equation can also be written in forms which emphasize the role of the "potential buoyancy" $b - \frac{N^2}{g}P$ or the sound speed $\overline{c}_s{}^2$:

$$-\frac{1}{g}\frac{\partial}{\partial t}(b - \frac{N^2}{g}P) + \frac{1}{\overline{c}_s{}^2}\frac{\partial}{\partial t}P + \nabla \cdot \mathbf{u} + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}(\overline{\rho}w) = 0 \qquad (f.3)$$

or

$$\frac{1}{\overline{c}_s{}^2}\frac{\partial}{\partial t}P - \frac{g}{\overline{c}_s{}^2}w + \nabla \cdot \mathbf{u} + \frac{\partial}{\partial z}w = 0 \qquad (f.3)$$

Hydrostatic case

When the motions are hydrostatic,

$$b = \frac{\partial P}{\partial z}$$

and

$$w = \left[\frac{1}{g} - \frac{1}{N^2}\frac{\partial}{\partial z}\right]\frac{\partial P}{\partial t}$$

When we substitute this into the mass equation, we find the

$$-\frac{1}{g}P_{zt} - \frac{1}{g}\frac{\overline{\rho}_z}{\overline{\rho}}P_t + \frac{1}{g}\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}(\overline{\rho}P_t) + \nabla \cdot \mathbf{u} - \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial}{\partial z}P_t = 0$$

so that

$$\nabla \cdot \mathbf{u} - \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial}{\partial z}P_t = 0$$

Since the horizontal equations have no coefficients depending on z, we can separate variables

$$\mathbf{u} = \mathbf{u}(x, y, t)F(z) \quad , \qquad P = P(x, y, t)F(z)$$

and still have

$$\frac{\partial}{\partial t}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla P \qquad (h.1)$$

In the mass conservation/ thermodynamic eqn., we now have

$$\nabla \cdot \mathbf{u}F(z) - \frac{\partial P}{\partial t}\left[\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial}{\partial z}F(z)\right] = 0$$

which will hold when

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial}{\partial z}F = -\frac{1}{gH_{eq}}F \qquad (h.2)$$

where we've introduced a notation for the separation constant which makes the horizontal part

$$\frac{\partial}{\partial t}P + gH_{eq}\nabla \cdot \mathbf{u} = 0 \qquad (h.3)$$

look very familiar: equations (h.1) and (h.3) are just the shallow-water equations with an "equivalent depth" H_{eq}.

If we have solid boundaries at $z = 0,\ H$, then $w = 0$ which implies

$$\frac{\partial}{\partial z}F = \frac{N^2}{g}F \quad at \quad z = 0,\ H \qquad (h.4 - fixed)$$

A free surface, on the other hand has $w = \frac{\partial}{\partial t}\eta$ with $\overline{P}(\eta) + \overline{\rho}(0)P(0) = 0 \quad \Rightarrow \quad P(0) = g\eta$ so that

$$\left[\frac{1}{g} - \frac{1}{N^2}\frac{\partial}{\partial z}\right]\frac{\partial P}{\partial t} = \frac{1}{g}\frac{\partial P}{\partial t} \quad \Rightarrow \quad \frac{\partial}{\partial z}F = 0 \qquad (h.4 - free)$$

Equations (h.2) and (h.4) give a Sturm-Liouville problem with a discrete set of eigenvalues H_{eq} (at least for the system with two boundaries).

Non-hydrostatic case

Let us now separate the vertical and horizontal parts of the full equations (f1-4). To do this, we need to assume a single frequency so that we can solve for w and b in terms of P. From the thermodynamic and vertical momentum equations, we find

$$\frac{\partial^2 w}{\partial t^2} + N^2 w = -P_{zt} + \frac{N^2}{g} P_t \quad \Rightarrow \quad w = \left[\frac{N^2}{g(N^2 - \omega^2)} - \frac{1}{N^2 - \omega^2} \frac{\partial}{\partial z} \right] P_t$$

which reduces to the hydrostatic case when $\omega^2 << N^2$. Likewise the buoyancy satisfies

$$b = \frac{N^2}{N^2 - \omega^2} \left[\frac{\partial}{\partial z} - \frac{\omega^2}{g} \right] P$$

With these forms, the conservation of mass equation looks like

$$\left[-\frac{1}{g} \frac{N^2}{N^2 - \omega^2} \left(\frac{\partial}{\partial z} - \frac{\omega^2}{g} \right) - \frac{1}{g} \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho}}{\partial z} + \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \frac{\overline{\rho} N^2}{g(N^2 - \omega^2)} - \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \frac{\overline{\rho}}{N^2 - \omega^2} \frac{\partial}{\partial z} \right] \frac{\partial P}{\partial t} + \nabla \cdot \mathbf{u} = 0$$

or

$$\left[\frac{1}{g} \frac{\partial}{\partial z} \left(\frac{\omega^2}{N^2 - \omega^2} \right) - \frac{\omega^2}{\overline{c_s}^2 (N^2 - \omega^2)} - \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \frac{\overline{\rho}}{N^2 - \omega^2} \frac{\partial}{\partial z} \right] \frac{\partial P}{\partial t} + \nabla \cdot \mathbf{u} = 0$$

The horizontal velocities and dynamic pressure P can still have the form $\mathbf{u} = \mathbf{u}(x, y, t) F(z)$, $P = P(x, y, t) F(z)$. But now the vertical structure equation becomes

$$\left[\frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \frac{\overline{\rho}}{N^2 - \omega^2} \frac{\partial}{\partial z} - \frac{1}{g} \frac{\partial}{\partial z} \left(\frac{\omega^2}{N^2 - \omega^2} \right) + \frac{\omega^2}{\overline{c_s}^2 (N^2 - \omega^2)} \right] F = -\frac{1}{g H_{eq}} F \qquad (VSE)$$

and the separation constant depends on the wave frequency. The boundary conditions become

$$\left[\frac{\partial}{\partial z} - \frac{N^2}{g} \right] F = 0 \qquad (Solid\ B)$$

or

$$\left[\frac{\partial}{\partial z} - \frac{\omega^2}{g} \right] F = 0 \qquad (Free\ B)$$

The horizontal structures still satisfy the Laplace tidal equations

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u} + f \hat{\mathbf{k}} \times \mathbf{u} &= -\nabla P \\ \frac{\partial}{\partial t} P + g H_{eq} \nabla \cdot \mathbf{u} &= 0 \end{aligned} \qquad (LTE)$$