# Laplace Tidal Equations — horizontal structure

## Equations

After separation of variables of the original linearized, hydrostatic, pressure coordinate equations (with the "standard approximation" of neglecting the horizontal part of the rotation) are

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{f} \times \mathbf{u} = -grad\ \Phi$$

$$\frac{\partial}{\partial t}\Phi + gH_{eq}\ div\ \mathbf{u} = 0$$

Here $\mathbf{f}$ is $f(\varphi)\hat{\mathbf{r}}$. The vertical structure of the horizontal velocities satisfies the equation

$$\frac{\partial}{\partial p}\frac{1}{S}\frac{\partial}{\partial p}F = -\frac{1}{gH_{eq}}F$$

with $S = -\Pi\ \overline{\theta}_p$. The solution to this, with suitable boundary conditions, defines the equivalent depths $H_{eq}$.

## Solution of the horizontal eqns.

If we take $\mathbf{f}\times$ the momentum equations and subtract it from $\partial/\partial t$ of the same equations, we find

$$\frac{\partial^2}{\partial t^2}\mathbf{u} + f^2\mathbf{u} = -grad\ \frac{\partial}{\partial t}\Phi + \mathbf{f} \times grad\ \Phi$$

For motions with frequency $\omega$, the velocity is given by

$$\mathbf{u} = -\frac{1}{f^2 - \omega^2}grad\ \frac{\partial}{\partial t}\Phi + \frac{\mathbf{f}}{f^2 - \omega^2} \times grad\ \Phi$$

Taking the divergence and substituting in the mass equation yields

$$\frac{\partial}{\partial t}\Phi - gH_{eq}div\frac{1}{f^2 - \omega^2}grad\ \frac{\partial}{\partial t}\Phi + gH_{eq}div\left[\frac{\mathbf{f}}{f^2 - \omega^2} \times grad\ \Phi\right] = 0$$

Using the vector identity

$$curl\ \Phi\mathbf{v} = grad\ \Phi \times \mathbf{v} + \Phi curl\ \mathbf{v}$$

gives

$$\frac{\partial}{\partial t}\Phi - gH_{eq}div\frac{1}{f^2 - \omega^2}grad\ \frac{\partial}{\partial t}\Phi + gH_{eq}div\left[\Phi curl\ \frac{\mathbf{f}}{f^2 - \omega^2}\right] = 0$$

For the spherical case, with $f = 2\Omega\sin\varphi$, the last term simplifies to

$$gH_{eq}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\Phi\frac{1}{a}\frac{\partial}{\partial\varphi}\frac{f}{f^2 - \omega^2}\right]$$

If we put in $\Phi \sim \exp(\imath m\lambda - \imath\omega t)$, we end up with a single equation for the meridional structure, with the frequency acting somewhat like the eigenvalue.

*Small scale, high frequency*

If we consider small scale waves in the sense that $grad\ \Phi/\Phi >> grad\ f/f$, we can drop the last term and pull the $f^2 - \omega^2$ out to find

$$\left[1 - \frac{gH_{eq}}{f^2 - \omega^2}\nabla^2\right]\Phi_t = 0$$

which has a zero frequency mode and a mode with

$$\omega^2 = f^2 + gH_{eq}\mathbf{k}^2$$

where the wavenumber is defined by the solutions of $\nabla^2\Phi = -\mathbf{k}^2\Phi$ as in the barotropic problem. These are the gravity waves.

*Low frequency*

The other mode has a low frequency (at least in the small scale limit) and satisfies

$$\frac{\partial}{\partial t}\Phi - gH_{eq}div\frac{1}{f^2}grad\ \frac{\partial}{\partial t}\Phi - gH_{eq}\frac{1}{a\cos\varphi}\frac{\beta}{f^2}\frac{\partial}{\partial\lambda}\Phi = 0$$

(at least away from the equator). In the case where the scales are still small, we can approximate this by

$$\frac{\partial}{\partial t}\left[\nabla^2 - \frac{f^2}{gH_{eq}}\right]\Phi + \beta\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\Phi = 0$$

a form of the Rossby wave equation. The local phase speed $c = a\cos\varphi(\omega/m)$ is given by

$$c = -\frac{\beta}{\mathbf{k}^2 + f^2/gH_{eq}}$$