Laplace Tidal Equations — Vertical Structure

The vertical structure is given by the solution of

$$\overline{\alpha}\frac{\partial}{\partial z}\frac{1}{\overline{\alpha}(N^2-\omega^2)}\frac{\partial}{\partial z}F + \left[\frac{\omega^2}{\overline{c}_s{}^2(N^2-\omega^2)} - \frac{\partial}{\partial z}\frac{\omega^2}{g(N^2-\omega^2)}\right]F = -\frac{1}{gH_{eq}}F$$

with

$$(\frac{\partial}{\partial z} - \frac{N^2}{g})F = 0 \qquad (FixedB)$$

and/or

$$(\frac{\partial}{\partial z} - \frac{\omega^2}{g})F = 0 \qquad (FreeB)$$

Low frequency

When ω is small, the VSE simplifies to

$$\overline{\alpha}\frac{\partial}{\partial z}\frac{1}{\overline{\alpha}N^2}\frac{\partial}{\partial z}F = -\frac{1}{gH_{eq}}F$$

with

$$(\frac{\partial}{\partial z} - \frac{N^2}{g})F = 0 \qquad (FixedB)$$

and/or

$$\frac{\partial}{\partial z}F = 0 \qquad (FreeB)$$

In the case where $N^2H << g$, the boundary conditions become $F_z = 0$ at both types of surfaces. This equation clearly has a barotropic solution $F = 1$ with infinite equivalent depth. If we don't drop the N^2/g term, we can estimate H_{eq} by using $F \simeq 1 + f$ and integrating from a rigid bottom at $z = 0$ to a free surface at $z = H$. We find

$$H_{eq} = H\left[\frac{1}{H\overline{\rho}(0)}\int_0^H \overline{\rho}(z)dz\right] \simeq H$$

combining this with the horizontal equations gives us the dispersion relationship for long surface gravity waves

$$\omega^2 = f^2 + gH\mathbf{k}^2$$

and barotropic Rossby waves

$$\omega = -\frac{\beta k}{\mathbf{k}^2 + f^2/gH}$$

Generally, we deal with scales small compared to the external deformation radius $\sqrt{gH}/f$ and can just use

$$\omega = -\frac{\beta k}{\mathbf{k}^2}$$

In addition, we have a set of internal modes. If we use a WKB approximation so that $F \simeq \exp(\imath mz)$, we have

$$gH_{eq} \simeq N^2/m^2$$

and

$$\omega^2 = \frac{f^2m^2 + N^2\mathbf{k}^2}{m^2}$$

(long internal gravity waves) or

$$\omega = -\frac{\beta k}{\mathbf{k}^2 + m^2f^2/N^2}$$

(baroclinic Rossby waves).

Intermediate frequencies

When the frequency is comparable to N or f and $N^2H << g$, $N^2H^2 << \overline{c}_s{}^2$ we have

$$gH_{eq} \simeq (N^2 - \omega^2)/m^2$$

for the internal modes, leading to the full internal wave relationship

$$\omega^2 = \frac{f^2m^2 + N^2\mathbf{k}^2}{\mathbf{k}^2 + m^2}$$

High frequencies

When the frequencies are large, the vertical scales will be short and the VSE simplifies to

$$\frac{\partial^2}{\partial z^2}F + \frac{\omega^2}{\overline{c}_s{}^2}F = \frac{\omega^2}{gH_{eq}}F$$

with

$$\frac{\partial}{\partial z}F = 0 \qquad (FixedB)$$

and/or

$$(\frac{\partial}{\partial z} - \frac{\omega^2}{g})F = 0 \qquad (FreeB)$$

For the internal modes,

$$gH_{eq} = \frac{\omega^2\overline{c}_s{}^2}{\omega^2 - m^2\overline{c}_s{}^2}$$

and the high–frequency horizontal equation gives

$$\omega^2 = gH_{eq}\mathbf{k}^2 = \frac{\omega^2\overline{c}_s{}^2\mathbf{k}^2}{\omega^2 - m^2\overline{c}_s{}^2}$$

or

$$\omega^2 = \overline{c}_s{}^2(\mathbf{k}^2 + m^2)$$

—sound waves.

The external mode has $F \sim exp(mz)$ with $m = \omega^2/g$ from the free surface boundary condition. The equivalent depth is

$$gH_{eq} = \frac{\omega^2 \overline{c}_s{}^2}{\omega^2 + m^2 \overline{c}_s{}^2}$$

Since these waves are still slow compared to the sound speed, this simplifies to

$$gH_{eq} = \frac{\omega^2}{m^2} = \frac{\omega^2 g^2}{\omega^4} = \frac{g^2}{\omega^2}$$

and the horizontal equation gives

$$\omega^2 = gH_{eq}\mathbf{k}^2 = \frac{g^2}{\omega^2}\mathbf{k}^2 \quad \Rightarrow \quad \omega^4 = g^2\mathbf{k}^2$$

or

$$\omega^2 = g|\mathbf{k}|$$

the dispersion relationship for short gravity waves.