Example of Curvilinear Coordinates

We consider the case of polar coordinates. The coordinate functions are

$$\begin{aligned}
\xi_1 &= (x^2 + y^2)^{(1/2)} \equiv r \\
\xi_2 &= atan(y/x) \equiv \theta \\
\xi_3 &= z
\end{aligned}$$

So that

$$\frac{\partial}{\partial x_i}\xi_j = \begin{pmatrix} x/r & y/r & 0 \\ -y/r^2 & x/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the scale factors are the inverses of the lengths of each of the row vectors:

$$\begin{aligned}
h_1 &= 1 \\
h_2 &= r \\
h_3 &= 1
\end{aligned}$$

The transformation matrix is given by

$$\begin{aligned}
\gamma_{ij} = h_i \frac{\partial}{\partial x_j}\xi_i &= \begin{pmatrix} x/r & y/r & 0 \\ -y/r & x/r & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

From these relationships, we find the gradient

$$(grad\ \phi) = \frac{1}{h_i}\frac{\partial}{\partial \xi_i}\phi = \begin{pmatrix} \frac{\partial}{\partial r}\phi \\ \frac{1}{r}\frac{\partial}{\partial\theta}\phi \\ \frac{\partial}{\partial z}\phi \end{pmatrix}$$

The divergence can be written as

$$\begin{aligned}
div\ \mathbf{F} &= \gamma_{ji}\frac{1}{h_j}\frac{\partial}{\partial \xi_j}\gamma_{mi}F'_m \\
&= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F'_r \\ F'_\theta \\ F'_z \end{pmatrix} \\
&= \frac{1}{r}\frac{\partial}{\partial r}(rF'_r) + \frac{1}{r}\frac{\partial}{\partial\theta}F'_\theta + \frac{\partial}{\partial z}F'_z
\end{aligned}$$

You can also obtain this from the formula

$$div\ \mathbf{F} = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial\xi_j}\left(\frac{h_1h_2h_3}{h_j}F'_j\right) = \frac{1}{r}\frac{\partial}{\partial\xi_j}\left(\frac{r}{h_j}F'_j\right)$$
$$= \frac{1}{r}\frac{\partial}{\partial r}(rF'_r) + \frac{1}{r}\frac{\partial}{\partial\theta}F'_\theta + \frac{1}{r}\frac{\partial}{\partial z}(rF'_z)$$

The curl can be found from

$$curl\ \mathbf{F} = \gamma_{il}\epsilon_{ljk}\gamma_{mj}\frac{1}{h_m}\frac{\partial}{\partial\xi_m}\gamma_{nk}F'_n =$$

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\left[\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} F'_r \\ F'_\theta \\ F'_z \end{pmatrix}\right]$$
$$= \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial\theta}F'_z - \frac{\partial}{\partial z}F'_\theta \\ \frac{\partial}{\partial z}F'_r - \frac{\partial}{\partial r}F'_z \\ \frac{\partial}{\partial r}F'_\theta + \frac{1}{r}F'_\theta - \frac{1}{r}\frac{\partial}{\partial\theta}F'_r \end{pmatrix}$$

Again this can be obtained from the formula

$$curl\ \mathbf{F} = \epsilon_{ijk}\frac{1}{h_jh_k}\frac{\partial}{\partial\xi_j}(h_kF'_k) = \epsilon_{ijk}\frac{h_i}{h_1h_2h_3}\frac{\partial}{\partial\xi_j}(h_kF'_k)$$

The transformed accelerations look like

$$\mathbf{A} = \frac{\partial}{\partial t}u'_i + \gamma_{ik}\gamma_{mj}u'_m\gamma_{nj}\frac{1}{h_n}\frac{\partial}{\partial\xi_n}\gamma_{sk}u'_s$$
$$= \frac{\partial}{\partial t}u'_i + \gamma_{ik}u'_j\frac{1}{h_j}\frac{\partial}{\partial\xi_j}\gamma_{sk}u'_s$$
$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}(u'\frac{\partial}{\partial r} + \frac{v'}{r}\frac{\partial}{\partial\theta} + w'\frac{\partial}{\partial z})\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}$$

Using a product rule and working through the derivatives gives

$$\mathbf{A} = \frac{\partial}{\partial t}u'_i + \begin{pmatrix} (u'\frac{\partial}{\partial r} + \frac{v'}{r}\frac{\partial}{\partial\theta} + w'\frac{\partial}{\partial z})u' - \frac{v'^2}{r} \\ (u'\frac{\partial}{\partial r} + \frac{v'}{r}\frac{\partial}{\partial\theta} + w'\frac{\partial}{\partial z})v' + \frac{u'v'}{r} \\ (u'\frac{\partial}{\partial r} + \frac{v'}{r}\frac{\partial}{\partial\theta} + w'\frac{\partial}{\partial z})w' \end{pmatrix}$$

If we use the vector form

$$\mathbf{A} = \frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial t}\mathbf{u} + grad\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2}\right) - \mathbf{u} \times curl\ \mathbf{u}$$
$$= \frac{\partial}{\partial t}u'_i + u'_j\frac{1}{h_j}\frac{\partial}{\partial\xi_j}u'_i + \frac{u'_i}{h_i}u'_j\frac{1}{h_j}\frac{\partial}{\partial\xi_j}h_i - \frac{u'_ju'_j}{h_j}\frac{1}{h_i}\frac{\partial}{\partial\xi_i}h_j$$
$$= \frac{\partial}{\partial t}u'_i + u'_j\frac{1}{h_j}\frac{\partial}{\partial\xi_j}u'_i + u'_iu'_j\frac{1}{h_j}\frac{\partial}{\partial\xi_j}\log h_i - u'_ju'_j\frac{1}{h_i}\frac{\partial}{\partial\xi_i}\log h_j$$

we end up at the same point fairly easily.