# 12.843 – project 1

## 1) Barotropic vorticity eqn. numerical lab

*Equations:*

We can solve the equations for homogeneous, incompressible, two–dimensional flow

$$\frac{d}{dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\frac{p}{\rho} + \nu\nabla^2\mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

using a streamfunction

$$\mathbf{u} = -\nabla \times \hat{\mathbf{k}}\psi(x, y, t) \qquad \text{or} \quad u = -\frac{\partial}{\partial y}\psi \ , \quad v = \frac{\partial}{\partial x}\psi$$

From the divergence of the momentum equations, we find

$$\nabla^2\frac{p}{\rho} = \nabla \cdot f\nabla\psi - \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$$

so, given $\psi$, we can find the velocities and the pressure $p/\rho$. From the curl of the momentum equations, we find the vorticity equation

$$\frac{d}{dt}q = \nu\nabla^2 q$$

with

$$q = f + \hat{\mathbf{k}} \cdot \nabla \times \mathbf{u} = f + \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = f + \nabla^2\psi$$

Given $\psi$, we can find $q$ and then evaluate the advection and diffusion terms to step $q$ forward in time. Inverting the Laplace operator allows us to calculate $\psi$ at the new time.

For these experiments, we use a doubly-periodic model — flow out the left comes back in the right and flow out the bottom comes back in the top. The model can do evolution calculations, but the main focus is on the relationship between PV anolmalies and flow patterns.

*Problems:*

GEOSTROPHY: Given the initial $\psi(x, y, 0)$ field, you can then calculate the other fields and display them. You can also watch the advancement in time of the streamfunction.

- Consider a single vortex

$$\psi = U_0 L \exp[\frac{1}{2} - \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{2}\left(\frac{y}{L}\right)^2]$$

  (or equivalently) $\psi = U_0 L * \texttt{gauss}(x/L, y/L)$. How do $\Delta\psi$ and $\Delta(p/\rho)$ compare for different $U_0$, $L$ values? In a geostrophic state, these would be related by $f\psi = \frac{p}{\rho}$. Calculate the pressure analytically using polar coordinates. How big are the ageostrophic terms and how does the sign of the streamfunction affect the relationship? How does this evolve when $\beta = df/dy$ is not zero? How geostrophic is the flow later on?

- Consider a single vortex in a shear flow

$$\psi = U_0 L * \texttt{gauss}(x/L, y/L) + U_s W \cos(2\pi y/W)$$

  What happens for different strengths and signs of the vortex/ shear? You can also try different $y$ locations and sizes. Note — the shear field must be periodic over the north-south extent of the domain.

- Consider a vortex on the $\beta$-plane

$$q - f = U_0/L * \texttt{gauss}(x/L, y/L)$$

  Discuss the evolution. How geostrophic is the flow at later times?

PV INVERSION: The model allows you to specify $q' = q - \beta y$ and then calculate other fields. Or you can specify $\psi$ and calculate $q$.

- What is the streamfunction for a circular patch of uniform vorticity? Numerically, you'll want to smooth out the edge; you can use `exp(-(x.^2+y.^2).^4)`. What about the periodic boundaries?
- What does $q$ look like for an isolated streamfunction (meaning the velocities fall off faster than $1/r$; e.g. the gaussian form above)?
- Consider an elliptical eddy in $q$; how does the streamfunction differ and what does this say about the evolution?
- Consider two vortices with (a) the same sign or (b) opposite signs. Describe the streamfunction and what it suggests about the evolution.

**2) Gas giant**

Suppose you model a gas giant as a rotating, isentropic sphere (or ellipsoid) of gas. Convince yourself that the thermodynamics implies $\rho$ can be written as $\rho(p, s)$ and, in this situation, is just a function of $p$. Write the momentum equations in terms of $P$ where $dP = dp/\rho(p)$. Show that zonal flows with $\vec{\Omega} \cdot \nabla u = 0$ are solutions to the equations. Suppose you have a jet like this a mid-latitudes; what does the pressure field $p$ look like? Assume that the gas is self-gravitating so that $\rho$ increases towards the center. Make a qualitative sketch.

**3) Rossby waves**

Consider a Rossby wave on the $\beta$-plane

$$\psi = A \cos(kx + \ell y - \omega t)$$

1) Show from the PV equation that this is an exact nonlinear solution and find $\omega$
2) Consider the momentum equations for the flow parallel to $\mathbf{k} = (k, \ell)$ and perpendicular to $\mathbf{k}$. Show that one is geostrophically balanced and find the pressure field. Use this to find the the acceleration in the other momentum equation. Use this to calculate $\omega$ and show it is the same as above.