# 12.843 - project 2

## Derivations

*Shear layer*

1a) Shear layer instability: consider

$$q = \zeta_0 + \Delta_1 \mathcal{H}(y - y_1 - \eta_1(x,t)) + \Delta_2 \mathcal{H}(y - y_2 - \eta_2(x,t))$$

Let $y_1 = -W/2$ and $y_2 = W/2$. Linearize, assuming $\eta_j = n_j \exp(\imath k[x - ct])$ and $n_j$ is suitably small. Write the matrix eigenvalue equation for the $n$'s.

1b) Rayleigh theorem: Show that instability requires $\Delta_1 \Delta_2 < 0$. Hint: consider

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = c \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

and figure out the constraints on $m_{21} m_{12}$ such that the imaginary part of $c$ could be nonzero.

1c) Fjortoft theorem: Show that you stabilize the flow when $U_i \Delta_i$ becomes negative.

1d Plot $c$ for $kW$ from near zero to 3. Plot the growth rate $k\,\mathrm{Im}(c)$.

*Adjustment*

2a) Consider an initial state

$$v = A \exp(-k|x|) \quad , \quad h = H \quad , \quad u = 0$$

From the linearized SW dynamics, find the final state assuming all the gravity waves radiate away. Evaluate the ratio of the geostrophic final state energy to the initial energy as a function of $kR_d$.

2b) Contrast this with the case

$$v = 0 \quad , \quad h = H + A \exp(-k|x|) \quad , \quad u = 0$$

2c) consider what might be a simple, qualitative way to understand the differences: how large a region is out of balance initially for the two cases and small and large $k$ values? This indicates how much mass needs to be moved and how much energy might be left for the steady state.

**Numerical labs**

*Frontal waves and shear instability*

From the derivation with $\zeta_0 = 0$, the flow is unstable for $kW$ smaller than 1.2785. Verify this is a good estimate from your analysis. Show that the numerical model ("shear layer") with $k = 1$, gives reasonable agreement with this (e.g., $W = 1.25$ is unstable, 1.3 is not). But we now want to look at the question of nonlinear stability: is, in fact, $W = 1.3$ stable as we increase the perturbation amplitude? A related question is whether the unstable mode just below the critical condition equilibrates at some amplitude which gets smaller as $kW$ approaches the value above?

I've also put up a code which computes the stability for a tanh shear layer on a $\beta$-plane. For $U = U_0 \tanh(y/W)$, what value of $\beta$ will stabilize the flow?

You can use the code (with $\beta = 0$) to see what happens as $W$ becomes less than the critical width, which, in this case is 1. Is the nonlinear evolution similar or different from the broken line profile? What happens for $W = 0.5$ as you increase $\beta$? Use the stability code to estimate the value for which the flow becomes stable. It should be

$$\frac{\beta W^2}{U_0} = \frac{4}{3\sqrt{3}}$$

with $U_0$ the maximum basic state velocity (1 in the code). How is it different with discrete wavenumbers?

*Adjustment*

Using the 2D SW code and inversion codes, examine the behavior of a dipolar initial condition

$$Q = \epsilon x \exp(-[x^2 + y^2]/2)$$

Try and monitor gravity wave energy by looking at a few points away from the vortex. Consider

2a) unbalanced initial conditions $[h' = -Q/\gamma^2(1 + Q),\ \psi = 0]$
2b) geostrophically balanced $[h' = \psi]$
2c) nonlinear balance

What happens for small and medium Rossby numbers (less than one, though)?