12.803 - Pset 3

1. Problem

a) Derive the two-layer model

$$\frac{\partial}{\partial t}q_1 + J(\psi_1, q_1) = 0$$
$$\frac{\partial}{\partial t}q_2 + J(\psi_2, q_2) = 0$$

with

$$q_1 = \nabla^2\psi_1 + F_1(\psi_2 - \psi_1) + \beta y \quad , \qquad q_2 = \nabla^2\psi_2 + F_2(\psi_1 - \psi_2) + \beta y$$

by starting with the QG voricity and buoyancy equations

$$\frac{D}{Dt}(\zeta + \beta y) = f_0 \frac{\partial}{\partial z} w \quad , \quad \frac{D}{Dt} b + N^2 w = 0$$

with

$$\zeta = \nabla^2 \psi \quad , \quad b = f_0 \frac{\partial}{\partial z}\psi \quad , \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \psi_x \frac{\partial}{\partial y} - \psi_y \frac{\partial}{\partial x}$$

Show that these give the ordinary Boussinesq QG eqns when you eliminate w.

b) Take N^2 as a delta function $N^2 = g'\delta(z - H_2)$ where $z = 0$ is the bottom, the interface between the two layers is centered at $z = H_2$ and $H_1 + H_2$ is the total depth. For the buoyancy eqn to be well-behaved, you need

$$\psi = \psi_2 + (\psi_1 - \psi_2) * (z > H_2)$$

so that b has a delta-function which can cancel the one in the vertical advection of the background stratification.

c) From this, $\frac{\partial}{\partial z} w$ is constant in each layer and therefore is just $\pm w(x, y, H_2, t)$ divided by the relevant layer depth.

d) Use this in the two vorticity equations, one for $\nabla^2\psi_1$ and one for $\nabla^2\psi_2$ to get the two-layer equations.

e) (Finally – to the problem) Let $q_j = \overline{Q}_{j_y} y + q_j' \exp(\imath kx - kct)$ and $\psi_j = -U_j y + \psi_j' \exp(\imath kx - kct)$. Write the matrix equation relating q_j' to ψ_j' and construct the eigenvalue equation for c and the eigenvector $\mathbf{q}'$

$$\mathbf{M}\mathbf{q}' = c\mathbf{q}'$$

Note the similarity to the Eady problem.

From the properties of $\mathbf{M}$ show that instability requires the two $\overline{Q}_y$ constants to be opposite sign. Show that this is sufficient for instability.

2. Derivations

2a) Nondimensionalize the β-plane SW equations

$$\frac{\partial}{\partial t}\mathbf{u} + (f+\zeta)\hat{\mathbf{z}} \times \mathbf{u} = -\nabla\left(gh' + \frac{1}{2}|\mathbf{u}|^2\right)$$
$$\frac{\partial}{\partial t}h + \nabla \cdot (H\mathbf{u} + h'\mathbf{u}) = 0$$

using a velocity scale U, length scale L, advective time scale, geostrophic estimate of h'.

2b) Write the nondimensional vorticity, divergence, and PV equations (both for q and for $Q = Hq - f_0$ from these.

2c) Write the lowest order in Rossby number for all these, assuming $\beta L/f_0 \sim \epsilon$ and $L \sim R_d$. Show that the lowest order has $\psi_0 = h_0'$ and $\varphi_0 = 0$. Show that the lowest order Q_0 equation is just the QG equation.

2d) Write the first order momentum, mass eqns. Solve for $\mathbf{u}_1$ and compute its divergence. Use geostrophy to get an evolution equation for the lowest order PV, all expressed in terms of h_0' or ψ_0.

2e) Write the first order divergence equation using $D_0 = 0$ as you've shown. The equation for h_1' in terms of ψ_0 should look familiar.

3. Numerical experiments

Using the SW code and inversion codes, examine the behavior of a dipolar initial condition

$$Q = \epsilon x \exp(-[x^2 + y^2]/2)$$

Try and monitor gravity wave energy by looking at a few points away from the vortex. Consider

3a) unbalanced initial conditions [$h' = -Q/\gamma^2(1+Q)$, $\psi = 0$]

3b) geostrophically balanced [$h' = \psi$]

3c) nonlinear balance

What happens for small and medium Rossby numbers (less than one, though)?