12.843 — Baroclinic Inversion/ Instability — Numerical Experiments

We're going to look at the 2 layer model and baroclinic instability including nonlinearity. The model solves

$$\left(\frac{\partial}{\partial t} + \frac{U}{2}\frac{\partial}{\partial x}\right) q_1 + Q_{1y}\frac{\partial}{\partial x}\psi_1 + J(\psi_1, q_1) = -rq_1$$
$$\left(\frac{\partial}{\partial t} - \frac{U}{2}\frac{\partial}{\partial x}\right) q_2 + Q_{2y}\frac{\partial}{\partial x}\psi_2 + J(\psi_2, q_2) = -rq_2$$

with

$$q_1 = \nabla^2\psi_1 + F_1(\psi_2 - \psi_1) \quad , \quad q_2 = \nabla^2\psi_2 + F_2(\psi_1 - \psi_2)$$

and

$$Q_{1y} = \beta + F_1 U \quad , \quad Q_{2y} = \beta - F_2 U$$

The domain will be a channel of width W and length $2W$. We nondimensionalize the equations using W for lengths and W/U for times and assume that the layer depths are equal $H_1 = H_2 = \frac{1}{2}H$. Then the equations look like the above except that U will be 1 (or zero in the absence of mean flow) and the parameters are

$$\beta \leftarrow \frac{\beta W^2}{U} \quad , \quad F_1,\ F_2 \leftarrow \frac{2f_0^2 W^2}{g' H}$$

There are two versions: a doubly period model and a periodic channel with NS walls (above). The doubly-periodic model has $r = 0$ and a filter to absorb fine-scale PV filaments that could otherwise cause numerical problems. The PV inversion is explicit in the doubly-periodic code; you'll want to contrast the nonlinear behavior in each one.

For the inversion, you specify the parameters F_1 and F_2; the evolution depends on U, and β as well.

Given the fields for q_1 and q_2 as functions of x and y, the program will calculate ψ and contour both the PV anomalies q_i and the full PV fields $q_i + [\beta + F_i(U_i - U_{3-i})]y$. It will also show the streamfunction anomalies ψ_i and the full streamfunction $\psi_i - U_i y$.

Once you have specified the PV and/or streamfunction fields, use the QG model to see how the flow evolves.

Derivations:

- Derive these equations as follows:
  - Start with the QG vorticity and buoyancy eqns

$$\frac{D}{Dt}(\zeta + \beta y) = f_0 \frac{\partial}{\partial z} w \quad , \quad \frac{D}{Dt} b + N^2 w = 0$$

  with

$$\zeta = \nabla^2 \psi \quad , \quad b = f_0 \frac{\partial}{\partial z} \psi \quad , \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \psi_x \frac{\partial}{\partial y} - \psi_y \frac{\partial}{\partial x}$$

  - show that these give the ordinary Boussinesq QG eqns when you eliminate w.
  - Think of N^2 as a delta function $N^2 = g'\delta(z - H_2)$ where $z = 0$ is the bottom, the interface between the two layers is centered at $z = H_2$ and $H_1 + H_2$ is the total depth. For the buoyancy eqn to be well-behaved, you need

$$\psi = \psi_2 + (\psi_1 - \psi_2) * (z > H_2)$$

  so that b has a delta-function which can cancel the one in the vertical advection of the background stratification.
  - Use this to rewrite the buoyancy eqn as

$$\frac{D}{Dt}(\psi_1 - \psi_2) + \frac{g'}{f_0} w = 0$$

  Show that it doesn't matter which ψ_j you use in the $\frac{D}{Dt}$.
  - From this, $\frac{\partial}{\partial z} w$ is constant in each layer and therefore is just $\pm w(x, y, H_2, t)$ divided by the relevant layer depth.
  - Use this in the two vorticity equations, one for $\nabla^2 \psi_1$ and one for $\nabla^2 \psi_2$ to get the two-layer equations (but for the total flow, not just the fluctuating part – i.e., the U_i terms will not appear).
- Suppose $q_1 = \exp(\imath k x)$ and $q_1 = 0$. Find ψ_1 and ψ_2. Comment on this and the opposite problem in terms of how the inversion looks.
- With the basic state U, show that the PV gradients can have opposite signs when there's enough shear. How does this depend on eastward vs. westward shear? Argue (qualitatively) that the physics of the instability, expressed in terms of inversion of PV anomalies and wave propagation/advection, is essentially similar to the Eady model.

Numerical experiments

· Explore the instability criterion using the numerics. How much shear do you need? Is the necessary condition sufficient? Show that you also need

$$k^2 + \ell^2 < F_1 + F_2$$

The channel code shows the case with $\ell = \pi$ – the $\sin(\ell y)$ mode which fits in the channel – and marks the k values which also fit $(n\pi)$.

· Show numerically that stable waves can still amplify, at least temporarily, if the initial phase relationships between upper and lower layers are correct.

· A growing plane wave is an exact solution to the equations above. What happens when such a wave is perturbed? Compare unperturbed to perturbed solutions.

· In the channel, what happens when only a single wave can grow? Look at the amplitude of the wave and the lower layer zonally averaged PV $yQ_{2y} + < q_2 >$ (since Q_{2y} is a constant) to explain qualitatively what happens.

· Consider a case with many unstable waves and describe briefly what happens.