Roadmap #1: BT Vorticity Eqn.

Cartesian tensors:

δ_{ij} and ϵ_{ijk}

$$\epsilon_{ijk} = \epsilon_{jki} = -\epsilon_{ikj}$$

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \quad *$$

$$(\nabla\phi)_i = \partial_i \phi$$

$$\nabla \cdot \mathbf{u} = \partial_i u_i \quad *$$

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k \quad *$$

$$(\nabla \times \mathbf{u})_i = \epsilon_{ijk} \partial_j u_k \quad *$$

Starred eqns use the summation convention: any index repeated on one side of the = sign but absent on the other is summed over.

- Prove $\nabla \times \nabla \times \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$ This is really a definition of the Laplacian acting on a vector. In Cart. coord., it is just the scalar Laplacian ($\nabla^2\phi = \nabla \cdot \nabla\phi$) acting on each component.
- For earth coords λ, θ, $z = r - a$ find $\nabla^2\phi$ and $\nabla^2\mathbf{u}$ with $\mathbf{u} = w\hat{z}$.

Helmholtz decomposition:

$$\mathbf{u} = -\nabla \times \Psi - \nabla\phi \quad \text{with} \quad \nabla \cdot \Psi = 0$$

- set $\mathbf{u} = \nabla^2 \mathbf{w}$ and use defn of vector ∇^2. How are ϕ and Ψ related to the divergence and vorticity if $\mathbf{u}$ is the velocity?
- 2D (x and y): Let $\mathbf{u} = \mathbf{u}_d + \mathbf{u}_c$ with

  $$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{u}_d \quad , \quad \nabla \times \mathbf{u} = \nabla \times \mathbf{u}_c$$

  and

  $$\phi = -\int^{\mathbf{x}} \mathbf{u}_d \cdot \hat{\mathbf{t}}\, ds \quad , \quad \psi = -\int^{\mathbf{x}} \mathbf{u}_c \cdot \hat{\mathbf{n}}\, ds$$

  Show these are well-defined (path independent) and that

  $$\mathbf{u}_d = -\nabla\phi \quad , \quad \mathbf{u}_c = -\nabla \times \psi\hat{\mathbf{z}} = \hat{\mathbf{z}} \times \nabla\psi$$

  and $\hat{\mathbf{z}} \cdot \nabla \times \mathbf{u} = \nabla^2\psi$
- What approx are necessary for these to hold on the sphere?

Rotation

If we write the transformation from an inertial frame $\mathbf{x}'$ from a rotating frame x as

$$\mathbf{x}' = \mathbf{R}(t)\mathbf{x} \quad , \quad \mathbf{R} = \begin{pmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then Newton's laws become

$$\mathbf{v}' = \mathbf{R}\mathbf{v} + \dot{\mathbf{R}}\mathbf{x} \quad , \quad \mathbf{a}' = \mathbf{R}\mathbf{a} + 2\dot{\mathbf{R}}\mathbf{v} + \ddot{\mathbf{R}}\mathbf{x} = \mathbf{f}'/m = \mathbf{R}\mathbf{f}/m$$

Therefore

$$\frac{D}{Dt}\mathbf{v} + 2\mathbf{R}^{-1}\dot{\mathbf{R}}\mathbf{v} + \mathbf{R}^{-1}\ddot{\mathbf{R}}\mathbf{x} = \mathbf{f}/m$$

- work out the matrices and show that

$$2\mathbf{R}^{-1}\dot{\mathbf{R}} = 2\Omega \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathbf{R}^{-1}\ddot{\mathbf{R}} = -\Omega^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathbf{R}^{-1}\ddot{\mathbf{R}}\mathbf{x} = -\nabla\frac{1}{2}\Omega^2(x^2+y^2)$$

Vorticity equation:

- Euler equations with $\boldsymbol{\zeta} = \nabla \times \mathbf{u}$ can be written

$$\frac{\partial}{\partial t}\mathbf{u} + (\boldsymbol{\zeta} + 2\Omega) \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\frac{|\mathbf{u}|^2}{2} - \nabla\Phi$$

  - for BT system $\rho = const$ or $\rho = \rho(p)$, $r.h.s. = -\nabla B$.
- write eqn for $\frac{\partial}{\partial t}\boldsymbol{\zeta}$ and for $\frac{\partial}{\partial t}\zeta_i$
- 2D – with $\mathbf{u} = \hat{\mathbf{z}} \times \nabla\psi$, rewrite the momentum equations and look at the divergence and divergence of $-\hat{\mathbf{z}} \times eqn$. Use $q = \zeta_3 + f$, $f = \hat{\mathbf{z}} \cdot 2\Omega$
- End result:
  - PV eqn

$$\frac{\partial}{\partial t}q + \mathbf{u}\cdot\nabla q = 0 \quad or \quad \frac{\partial}{\partial t}q + (\hat{\mathbf{z}} \times \nabla\psi)\cdot q = 0 \quad or \quad \frac{\partial}{\partial t}q + \hat{\mathbf{z}}\cdot(\nabla\psi \times \nabla q) = 0$$

  - Inversion

$$\nabla^2\psi = q - f$$

Inversion

- Greens' functions

$$\nabla^2 G(\mathbf{x}|\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$$

  or, in the absence of boundaries,

$$\nabla^2 G(\mathbf{x}-\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$$

  If

$$\nabla^2 \psi = \zeta$$

  then

$$\psi(\mathbf{x}) = \int d\mathbf{x}'\, G(\mathbf{x}-\mathbf{x}')\,\zeta(\mathbf{x}')$$

- BT form

$$G(\mathbf{x}) = \frac{1}{2\pi}\ln(\mathbf{x})$$

Point vortices

$$\zeta = s_i \delta(\mathbf{x}-\mathbf{X}_i(t))$$

- Show that

$$\frac{\partial}{\partial t}\mathbf{X}_i = s_j G'(|\mathbf{X}_i - \mathbf{X}_j|)\frac{1}{|\mathbf{X}_i - \mathbf{X}_j|}(-[Y_i - Y_j], [X_i - X_j]) = s_j G'(|\mathbf{X}_i - \mathbf{X}_j|)\frac{\hat{\mathbf{z}} \times (\mathbf{X}_i - \mathbf{X}_j)}{|\mathbf{X}_i - \mathbf{X}_j|}$$

  with $G'(r) = \frac{d}{dr}G(r)$
- Flow $V = s/2\pi r$

$$(u, v) = \frac{s}{2\pi r^2}(-y, x)$$

- dipoles

$$x(0) = 0 \quad , \quad y(0) = \pm d/2 \quad , \quad s = \pm S$$

- unequal strengths – show they rotate around a point between the vortices when they are the same sign and outside but on the line joining them when they are opposite signs.
- Inversion of a patch: suppose $\zeta = S\mathcal{H}(r_0 - r)$ with $\mathcal{H}$ the step function. Find the flow and streamfunction (solve in polar coords).

Rossby Waves

- Rossby waves - semigeostrophic viewpoint
  - Write the momentum equations on the $\beta - plane$ for a case where $\psi = \psi(x, t)$ (no y-dependence) using this to set the value of u.
  - Find the pressure; use this to simplify the $\frac{\partial}{\partial t}v$ equation
  - Look at the wave solutions and show that $\omega = -\beta/k$
  - Note the two kinds of balances of terms here
- Rossby waves on a sphere or beta plane
  - let $\zeta = -K^2\psi$; what does the BTVE become? What happens to the nonlinearity?
  - what is the dispersion relation?
  - what do solutions on the sphere look like?
- Rossby waves on a vorticity jump – we've done the circular problem, now consider the simpler zonal flow problem

$$\begin{aligned}
\zeta = \nabla^2\psi &= \zeta_0 + \Delta\mathcal{H}(y - \eta(x, t)) \\
\frac{\partial}{\partial t}\eta &= \frac{d}{dx}\psi(x, \eta(x, t), t) \\
&= \psi_y(x, \eta, t)\frac{\partial \eta}{\partial x} + \psi_x(x, \eta, t) \\
&\simeq \overline{\psi}_y(0)\frac{\partial \eta}{\partial x} + \psi'_x(x, 0, t)
\end{aligned}$$

- To solve, let's first write the more general form

$$\nabla^2\psi = \zeta_0 + \Delta_j\mathcal{H}(y - y_j - \eta_j)$$

where η_j will be taken to be sinusoidal with wavenumber k.

- Define

$$\left(\frac{\partial^2}{\partial y^2} - k^2\right) G_k(y-y') = \delta(y-y')$$

  and show that

$$\overline{u}(y) = -\int^y \zeta_0 - \Delta_j G_0(y-y_j)$$

$$\psi'(y) = -\Delta_j \eta_j G_k(y-y_j) \quad \Rightarrow$$

$$v'(y) = -G_k(y-y_j)\Delta_j \frac{\partial}{\partial x}\eta_j$$

- Now write the equation for $\frac{\partial}{\partial t}\eta_i$ Note that this is just a standard eigenvalue equation for waves of the form $\exp(k[x-ct])$.
- Solve for the single front

Shear instability

- Shear layer instability: solve when

$$\zeta_0 = 0 \quad , \quad \Delta_j = \mp 1 \quad , \quad y_i = \mp\frac{1}{2}$$

  What do these mean in terms of nondimensionalization?
- Rayleigh theorem: For the above problem, but with arbitrary Δ's, show that instability requires $\Delta_1\Delta_2 < 0$.
- Fjortoft theorem: Add ζ_0 and show that you stabilize the flow when $\overline{u}_i\Delta_i$ becomes negative.
- Derive the ordinary Rayleigh equation (with β) using $\overline{\mathbf{u}} = U$, $\overline{q} = Q$

$$(U-c)\left(\frac{\partial^2}{\partial y^2} - k^2\right)\psi + Q_y\psi = 0$$

  divide by $U-c$ assuming the flow is unstable so that $U \neq c$ anywhere, multiply by ψ^*, integrate over y and look at the imaginary and real parts. Use these to restate Rayleigh's and Fjortoft's thms.
- Arnold's Theorem:

$$\frac{\partial}{\partial t}q + J(\psi, q) = 0 \quad , \quad J(A,B) = \frac{\partial(A,B)}{\partial(x,y)} = det\begin{pmatrix} A_x & A_y \\ B_x & B_y \end{pmatrix}$$

  Basic state (could be 2D):

$$J(\Psi, Q) = 0 \quad \Rightarrow \quad \Psi = \Psi(Q)$$

  Perturbations:

$$\frac{\partial}{\partial t}q' + J(\Psi, q') + J(\psi', Q) = 0$$

Energy

$$\frac{\partial}{\partial t}\int \frac{1}{2}|\nabla\psi'|^2 = \int \psi'\Psi_Q J(Q,q')$$

Not quite enstrophy

$$\frac{\partial}{\partial t}\Psi_Q\frac{1}{2}q'^2 = -\int \Psi_Q q' J(\psi',Q) = -\int \psi'\Psi_Q J(Q,q')$$

$\Rightarrow$

$$\frac{\partial}{\partial t}\frac{1}{2}\int |\nabla\psi'|^2 + \Psi_Q q'^2 = 0$$

Stable if $\Psi_Q > 0$.

- Zonal flow. Translational symmetry implies we can take $\Psi = -\int^y U + \lambda y$. Then

$$\Psi_Q = -\frac{U-\lambda}{Q_y} \quad , \quad Q_y = \beta - U_{yy}$$

gives Rayleigh and Fjortoft criteria.

- Circular vortices $\Psi = \int^r V - \frac{1}{2}\lambda r^2$

$$\Psi_Q = \frac{V-\lambda r}{Q_r} \quad , \quad Q_r = \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rV$$

stable if Q_r is uniform in sign. If $Q_r > 0$ then $\frac{\partial}{\partial r}rV > 0$ implying a vortex with angular momentum increasing outwards is stable.

- waves on a patch

$$\zeta = S\mathcal{H}(r_0 + \eta(\theta,t) - r)$$

estimate ψ assuming η is very small:
  - you can Taylor-expand

$$\begin{aligned}\nabla^2\ \Psi &= S\mathcal{H}(r_0 - r)\\ \nabla^2\ \psi' &= S\Big[\mathcal{H}(r_0+\eta-r) - \mathcal{H}(r_0-r)\Big]\\ &\simeq S\eta\mathcal{H}'(r_0-r)\\ &= S\eta\delta(r_0-r)\end{aligned}$$

  Solve for $\eta = \eta_0\cos(m\theta - \omega t)$
  - the boundary is a material surface so that

$$\frac{\partial}{\partial t}\eta + v(r_0+\eta,\theta,t)\frac{1}{r_0+\eta}\frac{\partial}{\partial\theta}\eta = u(r_0+\eta,\theta,t)$$

  Linearize this and find ω. These are a kind of Rossby wave, travelling with the high PV fluid to the right (but advected by the background flow $V = \frac{\partial}{\partial r}\Psi$.

- Vortex instability: Use

$$q' = -Q_r \eta(r) e^{\imath m\theta - \imath\omega t} \quad \Rightarrow \quad \psi' = -\int dr' G_m(r,r') Q_r(r') \eta(r')$$

with the $e^{\imath m\theta - \imath\omega t}$ implicit

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r^2}\right] G_m(r,r') = \delta(r-r')$$

so that

$$-\imath\omega\eta(r) = -\imath m \frac{V}{r}\eta + \imath m \frac{1}{r}\int G(r,r') Q_r(r') \eta(r')$$

or

$$\frac{V}{r}\eta(r) - \frac{1}{r}\int G_m(r,r') Q_r(r') \eta(r') = \omega\eta(r)$$

We can look for $\exp(\imath m\theta)$, discretize in r and write the integral as a matrix operator; the equation becomes an ordinary matrix eigenvalue problem. We use the $\exp(\imath m\theta)$ form for G. Note that

$$\frac{1}{r}\frac{\partial}{\partial r} rV = Q \quad \Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} V - \frac{1}{r^2} V = Q_r \quad \Rightarrow \quad V(r) = \int G_1(r,r') Q_r(r')$$

So we can see something peculiar about $m = 1$: one solution is $\eta = 1$, $\omega = 0$

Beta drift

The evolution of an isolated vortex on the β-plane is complicated; we can gain some insight from the linear problem

$$\frac{\partial}{\partial t}\zeta = -\beta \frac{\partial}{\partial x}\psi \quad , \quad \nabla^2 \psi = \zeta$$

If we start with a $\zeta(x,y,0) = \zeta_0(r)$, then

$$\frac{\partial}{\partial x}\psi = \cos\theta \frac{\partial}{\partial r}\psi = V(r)\cos\theta \quad \textit{with} \quad V(r) = \frac{1}{r}\int_0^r r'\zeta_0(r')$$

For a patch $\zeta_0 = A\exp(-r^2/2L^2)$

$$V = \frac{A}{r}\left[1 - e^{-r^2/2L^2}\right]$$

Thus $\zeta_t = -\beta V \cos\theta$ shifts the vortex to the west, but also has larger-scale β-gyres, correxponding to a dipolar flow, with an anticyclone to the east and a cyclone to the west. These then bring in the nonlinear terms and induce northward acceleration. The algebra gets messy, but we can instead solve for the Taylor series in time numerically.

Matrix form and vorticity/Bernoulli

The Cartesian form of the advection is

$$u_j \frac{\partial u_i}{\partial x_j} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Later, we'll split the rate-of-strain matrix into symmetric and antisymmetric parts; for now, however, remove the antisymmetric part explicitly

$$\begin{aligned} u_j \frac{\partial u_i}{\partial x_j} &= \left[\begin{pmatrix} 0 & u_y - v_x & u_z - w_x \\ v_x - u_y & 0 & v_z - w_y \\ w_x - u_z & w_y - v_z & 0 \end{pmatrix} + \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} \right] \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ &= \left[\begin{pmatrix} 0 & -\zeta_3 & \zeta_2 \\ \zeta_3 & 0 & -\zeta_1 \\ -\zeta_2 & \zeta_1 & 0 \end{pmatrix} + \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} \right] \begin{pmatrix} u \\ v \\ w \end{pmatrix} \end{aligned}$$

The last term is just $\nabla[\frac{1}{2}|\mathbf{u}|^2$ while the first term is a rotation matrix around the vector $\boldsymbol{\zeta}$

$$\begin{pmatrix} 0 & -\zeta_3 & \zeta_2 \\ \zeta_3 & 0 & -\zeta_1 \\ -\zeta_2 & \zeta_1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\zeta_3 v + \zeta_2 w \\ \zeta_3 u - \zeta_1 w \\ -\zeta_2 u + \zeta_1 v \end{pmatrix} = \boldsymbol{\zeta} \times \mathbf{u}$$

Thus

$$u_j \frac{\partial u_i}{\partial x_j} = \boldsymbol{\zeta} \times \mathbf{u} + \nabla \frac{1}{2} |\mathbf{u}|^2$$