Roadmap #2: Three Dimensions: The Importance of Balance

Ertel potential vorticity (PV) theorem

• general form

$$q = \alpha \mathbf{Z} \cdot \nabla\theta \quad , \quad \frac{D}{Dt} q = 0$$

with $\mathbf{Z} = \boldsymbol{\zeta} + 2\Omega$ the absolute vorticity vector and $\alpha = 1/\rho$. Certain conditions have to apply.

- simplest standard derivation:
    1: write the product rule form for $\frac{\partial}{\partial t} q$
    2: for $\frac{\partial}{\partial t}\mathbf{Z}$, use the curl of

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{Z} \times \mathbf{u} = -\alpha\nabla p - \nabla\frac{1}{2}|\mathbf{u}|^2 - \nabla\Phi + \mathbf{F}$$

    with $\mathbf{F}$ representing non-conservative forces (friction). Do not expand $\nabla\times(\mathbf{Z}\times\mathbf{u})$. Instead, put this back in (step 1) and use the identity

$$\nabla\theta \cdot (\nabla \times \mathbf{A}) = \nabla \cdot (\mathbf{A} \times \nabla\theta) \quad \text{(prove)}$$

    Use the triple $\times$ rule for vectors on $(\mathbf{Z} \times \mathbf{u}) \times \nabla\theta$. Terms like $\mathbf{u} \cdot \nabla\theta$ and $\mathbf{Z} \cdot \nabla\theta$ are scalars – no particular coordinate trickery required. The other vector identity you need is $\nabla \cdot (\lambda\mathbf{A}) = \lambda\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\lambda$.
    3: For $\frac{\partial}{\partial t}\theta$, assume it's conserved except for some small term H

$$\frac{D}{Dt}\theta = H$$

    (θ could be the entropy; for an ideal gas the latter is $c_p \ln\theta$.)
    4: for $\frac{\partial}{\partial t}\alpha$ use the conservation of mass. Replace all $\mathbf{Z} \cdot \nabla\theta$ with q/α
    5: you should be able to cancel most terms now, ending up with $\frac{D}{Dt}q$ equaling various non-conservative terms and one involving the gradients of θ, p, and α.
    6: discuss why the latter term might vanish
- Derivation from the circulation thm (conservative form)
    1: Consider a material contour $\mathbf{X}(\lambda, t)$ on an surface of constant θ. Show that in an inertial frame

$$\frac{d}{dt}\oint \mathbf{u}' \cdot d\mathbf{X} = -\oint \alpha\nabla p \cdot d\mathbf{X}$$

    2: if $\alpha = \alpha(\theta, p)$ show the baroclinic torque term vanishes.
    3: use $\mathbf{u}' = \mathbf{u} + \Omega \times \mathbf{X}$ and Stokes' thm to write this in terms of the area enclosed by the contour, and $\mathbf{Z} \cdot \hat{\mathbf{n}}$. Note that $\hat{\mathbf{n}} = \nabla\theta/|\nabla\theta|$
    4: consider two neighboring theta surfaces and use conservation of mass to relate area changes to the thickness of the little cylinder made up of the boundary of the area (the contour) and the two surfaces: $1/|\nabla\theta$.
    5: put this back in step 3 and you're done.

Inversion

Although we have a conserved scalar, the inversion to determine the velocities is no longer possible. The PV depends on $\mathbf{u}$, θ, and ρ. Even for an incompressible fluid with $u = \nabla \times \Psi$ and $\zeta = \nabla^2 \Psi$ (when $\nabla \cdot \Psi = 0$), we still have two streamfunction variables and the other two all combined in q. We need some additional relationships to find the flow given q; these can be provided by balance equations – hydrostatic and geostrophic being the standard ones.

Hydrostatic balance

- why should the large-scale flow be hydrostatic? $\frac{\partial}{\partial z} p = -g\rho$ where $\nabla \Phi = g\hat{\mathbf{z}}$
- alternate vertical coords
  - change to (x, y, ξ, t) so that $z = z(x, y, \xi, t)$ is now a dependent variable.
  - Replace the vertical coordinate z by ξ. Show

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} - \frac{z_x}{z_\xi}\frac{\partial}{\partial \xi}$$

(likewise $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial y}$) and

$$\frac{\partial}{\partial z} \to \frac{1}{z_\xi}\frac{\partial}{\partial \xi}$$

The new vertical velocity is $\omega = \frac{D}{Dt}\xi$; show that the advection operator becomes

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial \xi}$$

The hydrostatic equation

$$\frac{\partial}{\partial z} p = -g\rho \quad \to \quad \frac{1}{\rho}\frac{\partial}{\partial \xi} p = -g z_\xi$$

so that the horizontal momentum eqns have

$$-\frac{1}{\rho}\nabla p \quad \to \quad -\frac{1}{\rho}\nabla p - \nabla g z$$

$$\frac{D}{Dt}\mathbf{u} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla \phi$$

  - The mass equation gives

$$\frac{D}{Dt}\ln\rho + \nabla \cdot \mathbf{u} - \frac{\nabla z \cdot \mathbf{u}_\xi}{z_\xi} + \frac{1}{z_\xi}\frac{\partial}{\partial \xi}\frac{D}{Dt}z = 0$$

or

$$\frac{D}{Dt}\ln\rho z_\xi + \nabla \cdot \mathbf{u} + \frac{\partial}{\partial \xi}\omega = 0$$

$$\frac{D}{Dt}\ln(-p_\xi)+\nabla\cdot\mathbf{u}+\frac{\partial}{\partial\xi}\omega=0 \quad \mathit{or} \quad \frac{\partial}{\partial t}h+\nabla\cdot h\mathbf{u}+\frac{\partial}{\partial\xi}h\omega=0$$

(h could be p_ξ). The thermodynamic equation is not changed

$$\frac{D}{Dt}\rho-\frac{1}{c_s^2}\frac{D}{Dt}p=0$$

PV

$$q=\frac{1}{h}[\nabla\times\mathbf{u}+\mathbf{f}]\cdot\nabla\theta$$

and we still need $\nabla\theta\cdot(\nabla\rho\times\nabla p)=0$.
Boundary conditions are messy though. The bottom is now at a time- and space-dependent value ξ_s such that

$$\phi(x,y,\xi_s,t)=gH(x,y)$$

The kinematic condition becomes

$$\omega(x,y,\xi_s,t)=\frac{\partial}{\partial t}\xi_s+\mathbf{u}(x,y,\xi_s,t)\cdot\nabla\xi_s$$

Commonly used forms

- pressure coordinates

$$\xi=p \quad\Rightarrow\quad \nabla p=0 \;\;,\;\; h=1$$

$$\frac{D}{Dt}\mathbf{u}+\mathbf{f}\times\mathbf{u}=-\nabla\phi$$

$$\frac{\partial}{\partial\xi}\phi=-\frac{1}{\rho}$$

$$\nabla\cdot\mathbf{u}+\frac{\partial}{\partial\xi}\omega=0$$

$$\frac{D}{Dt}\rho-\frac{\omega}{c_s^2}=0$$

For an ideal gas with

$$\frac{\theta}{\theta_0}=\frac{\rho_0}{\rho}\left(\frac{p}{p_0}\right)^{1/\gamma}$$

We can instead use

$$\frac{\partial}{\partial\xi}\phi=-G(p)\theta$$

$$\frac{D}{Dt}\theta=0$$

The PV is

$$q = (\nabla \times \mathbf{u} + \mathbf{f}) \cdot \nabla \theta$$

• ocean pressure coordinates

$$p = -\rho_0 g \xi \quad \Rightarrow \quad \nabla p = 0 \quad , \quad h = \rho_0 g$$

$$\frac{D}{Dt}\mathbf{u} + \mathbf{f} \times \mathbf{u} = -\nabla \phi$$

$$\frac{\partial}{\partial \xi}\phi = g\frac{\rho_0}{\rho} = b$$

$$\nabla \cdot \mathbf{u} + \frac{\partial}{\partial \xi}\omega = 0$$

$$\frac{D}{Dt}b = -\frac{g\rho_0}{\rho^2}\frac{D}{Dt}\rho = -\frac{g\rho_0}{\rho^2 c_s^2}\frac{D}{Dt}p = \omega\frac{\rho_0^2 g^2}{\rho^2 c_s^2} \simeq \omega\frac{g^2}{c_s^2}$$

and, if we regard c_s^2 as a function of ξ, we can redefine the buoyancy to include this term; its vertical derivative is just N^2 including the compressibility.

The PV is

$$q = (\nabla \times \mathbf{u} + \mathbf{f}) \cdot \nabla b$$

• pressure-like coords – let the vertical coordinate be $\xi(p)$, defined by some specified mean density state $\overline{\rho}(\xi)$ by

$$\frac{\partial p}{\partial \xi} \equiv -g\overline{\rho}$$

The momentum equations look like the hydrostatic Boussineq equations with p/ρ_0 replaced by $\phi = gz$ (remember z is a dependent variable, not a coordinate) and the buoyancy

$$\frac{\partial \phi}{\partial \xi} = b \equiv g\overline{\rho}/\rho$$

The mass equation is

$$\nabla \cdot \overline{\rho}\mathbf{u} + \frac{\partial}{\partial \xi}\overline{\rho}\omega = 0$$

The thermodynamic equation gives

$$\frac{\partial}{\partial t}b + \mathbf{u} \cdot \nabla b + \omega S = 0 \quad , \quad S = b_\xi - b\frac{\overline{\rho}_\xi}{\overline{\rho}} - \frac{b^2}{c_s^2} = \frac{\overline{\rho}^2}{\rho^2}N^2$$

Note that b is nearly g if ρ is close to $\overline{\rho}$, so the last two terms make sense. The PV is

$$q = \frac{1}{\rho}(\nabla \times \mathbf{u} + \mathbf{f}) \cdot \nabla \theta$$

The boundary conditions can be simplified if ξ_s does not vary too much: $\phi(\mathbf{x}, \xi_s(\mathbf{x}, t), t) = gH(\mathbf{x})$ becomes

$$\phi(x, y, 0, t) + g\frac{\overline{\rho}(0)}{\rho(x, y, 0, t)}\xi_s = gH$$

and

$$\omega(x, y, 0, t) = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)[\rho(x, y, 0, t)\overline{\rho}(0)(H - \phi(x, y, 0, t)/g)]$$

- isentropic coords.
  - Replace the vertical coordinate z by θ. Show

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} - \frac{z_x}{z_\theta}\frac{\partial}{\partial \theta}$$

(likewise $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial y}$) and

$$\frac{\partial}{\partial z} \to \frac{1}{z_\theta}\frac{\partial}{\partial \theta}$$

The new vertical velocity is $\omega = \frac{D}{Dt}\theta$; show that the advection operator becomes

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial \theta}$$

If θ is conserved $\omega = 0$, and advection becomes two-dimensional. We'll use that form in the following.
  - for an ideal gas

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla M$$

$M = c_pT + \Phi = \theta\Pi + \Phi$ with $\Pi = c_p(p/p_0)^{R/c_p}$.
  - use $w = \frac{D}{Dt}z$ in the mass equation to get

$$\frac{\partial}{\partial t}h + \nabla \cdot (\mathbf{u}h) = 0$$

with the thickness being $h = -\frac{\partial p}{\partial \theta}$.
  - PV: Show that

$$\frac{D}{Dt}q = 0 \quad , \quad q = \frac{\zeta + f}{h}$$

with $\zeta = \hat{\mathbf{z}} \cdot (\nabla \times \mathbf{u})$ ($v_x - u_y$ on the β-plane).
  - Show that

$$M_\theta = \Pi \quad \textit{and} \quad M_{\theta\theta} = -R\left(\frac{p}{p_0}\right)^{-1/\gamma} h$$

Shallow water system

The equations in isentropic coordinates, if discretized in θ, are simply a stack of shallow water systems (but with a more complicated relationship between the pressure and the height fields – in the Boussinesq case it's $M_{\theta\theta} = -h$ instead of $M = gh$). The equations are

$$\frac{\partial}{\partial t}\mathbf{u} + (\zeta + f)\hat{\mathbf{z}} \times \mathbf{u} = -\nabla(gh + \frac{1}{2}|\mathbf{u}|^2)$$
$$\frac{\partial}{\partial t}h + \nabla \cdot (h\mathbf{u}) = 0$$

• shallow water PV: show that

$$q = \frac{\zeta + f}{h}$$

is conserved. But we are going to subtract off the large, nearly constant part f_0/h_0 and multiply by h_0 to get a conserved property

$$Q = h_0 q - f_0 = \frac{h_0}{h}\left[\zeta + \beta y + f_0 \frac{h_0 - h}{h_0}\right]$$

• write the equations in terms of the streamfunction and potential

$$\mathbf{u} = \hat{\mathbf{z}} \times \nabla\psi - \nabla\varphi$$

Note that the PV involves only two of these: $\zeta = \nabla^2\psi$ and h/h_0. Inversion requires a link between these, relating h/h_0 to ψ. Time-stepping then needs an additional statement about ϕ.

• quasigeostrophy: we use the geostrophic relation for small rossby number

$$f_0\psi = g(h - h_0)$$

$h - h_0 \sim Ro\, h_0$ and $\nabla\varphi \sim Ro\, \nabla\psi$. Then

$$Q = \nabla^2\psi + \beta y - \frac{f_0^2}{gh_0}\psi \quad , \quad \frac{\partial}{\partial t}Q + J(\psi, Q) = 0$$

- formal approach: Nondimensionalize and expand in Ro assuming $Ro \sim \beta L/f_0 \sim (h - h_0)/h_0$.
- wave approach

Here, we build on the nature of linear solutions to the initial value problem to separate out and remove the gravity waves, leaving only the slow motions. The archetype is the "geostrophic adjustment" problem.

• geostrophic adjustment; separation into balanced and unbalanced flows: First, we'll solve the linear problem, then look briefly at the nonlinear system, and then use the insights to simplify the dynamics

- consider the linear equations for a 1-D system with constant f

$$\frac{\partial}{\partial t}u - fv = -\frac{\partial}{\partial x}gh'$$
$$\frac{\partial}{\partial t}v + fu = 0$$
$$\frac{\partial}{\partial t}h' + H\frac{\partial}{\partial x}u = 0$$

with initial conditions $u = 0$, $v = v_0(x)$, $h' = h'_0(x)$.

- show that
$$\frac{\partial}{\partial t}q = 0 \quad with \quad q = \frac{\partial}{\partial x}v - \frac{f}{H}h'$$
- in the divergence equation ($\frac{\partial}{\partial x}$ of the x-momentum), eliminate $\frac{\partial}{\partial x}v$ using the definition of the linear PV just above, and $\frac{\partial}{\partial t}u_x$ using the mass equation. The result is an equation for the thickness. Since $q(x,t) = q_0(x)$, you can write an equation for the steady part and the wavelike part.
- find the dispersion relation for the wave-like part
- Note that the steady part is entirely determined by the PV (invertable!) and the wavelike part has no PV signal.

• Nonlinear case: The simplest nonlinear problem starts with a box initial height field

$$h = h_0 * [-\frac{L}{2} < x < \frac{L}{2}]$$

We can look for a steady state solution, though there really is no way for the gravity waves to escape here. The edges will settle to a position $\pm R/2$. Thus

$$q = \frac{\zeta + f}{h} = \frac{f}{h_0}$$

But for steady flow, u must vanish and $v = gh_x/f$: it becomes geostrophically balanced. Or, defining $\psi_x = v$,

$$\nabla^2\psi = f\frac{h}{h_0} - f = \frac{f^2}{gh_0}\psi - f \quad \Rightarrow \quad \psi = \frac{gh_0}{f} + B\cosh(x/R_d)$$

$R_d = \sqrt{gh_0}/f$. Setting $\psi = 0$ at $R/2$ gives

$$B = \frac{-gh_0/f}{\cosh(R/2R_d)}$$

Setting the final mass anomaly to the initial one h_0L gives a transcendental equation for R

$$L = R - 2R_d\tanh(R/2R_d)$$

Asymptotically, $L = R - 2R_d$.

Next we start with a box initial height field but a non-zero height outside

$$h = h_0 + (h_1 - h_0) * [-\frac{L}{2} < x < \frac{L}{2}]$$

When all the gravity waves radiate away, the flow will become steady and the boundary between the two PV values will settle to positions $\pm R/2$. Thus

$$q = \frac{\zeta + f}{h} = \frac{f}{h_0} + \left(\frac{f}{h_1} - \frac{f}{h_0}\right) * [-\frac{R}{2} < x < \frac{R}{2}]$$

But for steady flow, u must vanish and $v = gh_x/f$: it becomes geostrophically balanced. Or, defining $\psi_x = v$, $h = h_0 + (f\psi/g)$. In the outer regions, then

$$\nabla^2\psi = \frac{f}{h_0}(h_0 + \frac{f\psi}{g}) - f = \frac{f^2}{gh_0}\psi \quad \Rightarrow \quad \psi = A\exp(-x/R_0)$$

with $R_0 = \sqrt{gh_0}/f$. In the inner region

$$\nabla^2\psi = \frac{f}{h_1}(h_0 + \frac{f\psi}{g}) - f = \frac{f^2}{gh_1}\psi - f\frac{h_1 - h_0}{h_1} \quad \Rightarrow \quad \psi = \frac{g(h_1 - h_0)}{f} + B\cosh(x/R_1)$$

$R_1 = \sqrt{gh_1}/f$. Matching ψ and $\frac{\partial}{\partial x}\psi$ at $R/2$ gives

$$B = \frac{-g\Delta/f}{\cosh(R/2R_1) + (R_0/R_1)\sinh(R/2R_1)} \quad , \quad Ae^{-R/2R_0} = -B(R_0/R_1)\sinh(R/2R_1)$$

Setting the final mass anomaly to the initial one ΔL gives a transcendental equation for R

$$L = \frac{RR_1\cosh(R/2R_1) + (2R_0^2 + RR_0 - 2R_1^2)\sinh(R/2R_1)}{R_1\cosh(R/2R_1) + R_0\sinh(R/2R_1)}$$

The radially symmetric problem does not seem to have analytic solutions: we have

$$\zeta = \begin{cases} \dfrac{fh'}{h_0} & r > R \\ \dfrac{f}{h_1}h' - f\dfrac{h_1 - h_0}{h_1} & r < R \end{cases}$$

with

$$h' = h - h_0 \quad , \quad fv + \frac{v^2}{r} = g\frac{\partial}{\partial r}h' \quad , \quad \zeta = \frac{1}{r}\frac{\partial}{\partial r}rv$$

The centrifugal term prevents an analytical solution, but it can be done numerically pretty easily using a shooting method. Once again, you solve for $v(r)$ and $h(r)$ then integrate the latter to find the L value corresponding to the R chosen.

The linear case corresponds to neglecting the centrifugal term and the difference between h_1 and h_0 in specifing the deformation radius. The

$$\nabla^2\psi - \frac{1}{R_d^2}\psi = -f\frac{h_1 - h_0}{h_1} * [r < L]$$

$R_d = f^2/gh_0$. This has constant + $I_0(r/R_d)$ solutions inside and $K_0(r/R_d)$ solutions outside. The latter implies that the net vorticity is zero. This does not mean the vortex is unstable: the PV gradient still has only one sign.

• slow adjustment: The previous problems suggest that gravity waves will radiate away leaving the PV mode to evolve slowly. Numerically, however, you either need some means of soaking them up or letting them leave the domain. For theoretical studies, though, you'd like to concentrate on the slow mode; in addition, there's the question of whether the shock represented by the initial conditions really is the appropriate representation of the large-scale motions. One way to approach an answer is to consider a slowly forced problem:

- use the linearized equations
- add mass by a source term

$$\frac{\partial}{\partial t}h + H\nabla \cdot \mathbf{u} = S(x) = \frac{s}{T}\cos(kx) * [0 < t < T]$$

- find the steady part of the flow for $t > T$
- calculate the energy in the steady flow

$$E = \frac{1}{2}\langle Hv^2 + gh^2\rangle$$

  with the average being over one spatial period $2\pi/k$.
- compare this to the total energy input

$$g\int_0^T \langle hS\rangle$$

  and show that most of the energy is in the steady flow when $T > 1/f$.
- you can do this in polar coords with $S \propto J_0(kr)$ instead – good exercise in special functions. You could also consider a localized source and look at the fraction of energy over the whole domain and over some finite area including the source.

• back to waves and QG: to find the equation for the slow mode evolution
- write the vorticity ζ and divergence D equations

$$\frac{\partial}{\partial t}\zeta + f_0 D = -(\zeta + \beta y)D - \mathbf{u}\cdot\nabla(\zeta + \beta y) = F_\zeta$$

$$\frac{d}{dt}D - f_0\zeta + \nabla^2 gh' = \beta y\zeta - \beta u + \frac{\partial}{\partial x}v\zeta - \frac{\partial}{\partial y}u\zeta - \nabla^2\frac{|\mathbf{u}|^2}{2} = F_D$$

$$\frac{\partial}{\partial t}h' + HD = -h'D - \mathbf{u}\cdot\nabla h' = F_h$$

- eliminate D from 1st and 3rd and rewrite 2nd in terms of $Q = \zeta - f_0 h'/H$

$$\frac{\partial}{\partial t}Q = F_\zeta - \frac{f_0}{H}F_h$$

$$\frac{d}{dt}D - f_0 Q + \nabla^2 gh' - \frac{f_0^2}{gH}gh' = F_D$$

$$\frac{\partial}{\partial t}h' + HD = F_h$$

- Split h' into the balanced part

$$(\nabla^2 - R_d^{-2})gh'_b = f_0 Q \quad or \quad \mathcal{L}h'_b = \frac{f_0}{g}Q \quad \Rightarrow \quad h'_b = \mathcal{L}^{-1}\frac{f_0}{g}Q$$

and the rest $h'_w = h' - h_b$

$$\frac{\partial}{\partial t}Q = F_\zeta - \frac{f_0}{H}F_h$$

$$\frac{d}{dt}D + \mathcal{L}gh'_w = F_D$$

$$\frac{\partial}{\partial t}h'_w + HD = F_h - \mathcal{L}^{-1}\frac{f_0}{g}(F_\zeta - \frac{f_0}{H}F_h)$$

- The slow evolution, to a first approximation says that h'_w and D are zero – the gravity waves have the two terms on the left of the 2nd and third balancing, but the time scales are much longer than GW periods, so that the $\frac{\partial}{\partial t}$ terms are small. Thus we can find $\frac{\partial}{\partial t}Q$ by evaluating the rhs of the 1st eqn under these circumstances. Since $D \sim 0$, the split of the velocities into ψ and ϕ

$$\mathbf{u} = \hat{\mathbf{z}} \times \nabla\psi - \nabla\phi = \hat{\mathbf{z}} \times \nabla\psi \quad since \quad \nabla^2\phi = -D$$

Then

$$F_\zeta = -J(\psi, \zeta + \beta y) \quad , \quad F_h = -J(\psi, h'_b) \quad and \quad Q = \nabla^2\psi - \frac{f_0}{H}h'_b$$

From the definition of h'_b, we see that $h'_b = \frac{f_0}{g}\psi$ and

$$\frac{\partial}{\partial t}Q = -J(\psi, Q + \beta y) \quad , \quad Q = \mathcal{L}\psi = [\nabla^2 - R_d^{-2}]\psi$$

- QG Rossby waves and inversion
  The QG equations are linear, so that superposition applies

$$\psi(\mathbf{x}) = \int d\mathbf{x}' G(|\mathbf{x} - \mathbf{x}'|)Q(\mathbf{x}')$$

The equivalent of a point vortex

$$(\nabla^2 - \gamma^2)G(\mathbf{x} - \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \quad , \quad \gamma = 1/R_d$$

is now shielded

$$G = -\frac{1}{2\pi}K_0(\gamma r) \quad , \quad \frac{\partial}{\partial r}G = \frac{\gamma}{2\pi}K_1(\gamma r)$$

– the velocity falls off as $\exp(-r)/\sqrt{r}$, the vorticity is negative around the point, and has net integral zero.

$$\zeta = \delta(\mathbf{x} - \mathbf{x}') - \frac{\gamma^2}{2\pi}K_0(\gamma r)$$

A finite area vortex

$$\psi = -Q_0\gamma^{-2}\begin{cases} 1 - \gamma R K_1(\gamma R) I_0(\gamma r) & r < R \\ \gamma R I_1(\gamma R) K_0(\gamma r) & r > R \end{cases}$$

has velocity

$$V = \frac{\partial}{\partial r}\psi = Q_0 R \begin{cases} K_1(\gamma R) I_1(\gamma r) & r < R \\ I_1(\gamma R) K_1(\gamma r) & r > R \end{cases}$$

and vorticity

$$\nabla^2\psi = Q_0 \gamma R \begin{cases} K_1(\gamma R) I_0(\gamma r) & r < R \\ I_1(\gamma R) K_0(\gamma r) & r > R \end{cases}$$

• Rossby waves:

$$Q' = (\nabla^2 - \gamma^2)\psi = -K^2\psi \quad \Rightarrow \quad J(\psi, Q) = J(\psi, -K^2\psi + \beta y) = J(\psi, \beta y) = \beta\frac{\partial}{\partial x}\psi = -\frac{\beta}{K^2}\frac{\partial}{\partial x}Q'$$

$$\frac{\partial}{\partial t}Q' - \frac{\beta}{K^2}\frac{\partial}{\partial x}Q' = 0 \quad \Rightarrow \quad c = -\frac{\beta}{K^2}$$

For plane waves, this gives

$$\omega = -\frac{\beta k}{k^2 + \ell^2 + \gamma^2}$$

Useful to write this as

$$\left(k + \frac{\beta}{2\omega}\right)^2 + \ell^2 = \left(\frac{\beta}{2\omega}\right)^2 - \gamma^2 \quad , \quad |\omega| \le \frac{\beta}{2\gamma} = -\frac{1}{2}\beta R_d \quad , \quad |c| \le \beta R_d^2$$

- reflection and transmission problems: these will have the same values of ω and of the tangential component of $\mathbf{k}$.

• balance and inversion; linear and higher order: the QG equations give an approximation to the slow evolution which again combines PV conservation (but now QGPV) and inversion. But we've had to take $h' << H$ and small Rossby number. What about nonlinear terms?

- Circular patch: The problem is

$$\frac{1}{r}\frac{\partial}{\partial r}rv - f_0\frac{h'}{H} = \frac{H+h'}{H}Q * [r<R]$$
$$\frac{v^2}{r} + f_0 v = g\frac{\partial}{\partial r}h'$$

with exterior PV f_0/H and interior value $(f_0+Q)/H$. Scale r by R, v by $|Q|R$ and h' by $f_0|Q|R^2/g$

$$\frac{1}{r}\frac{\partial}{\partial r}rv - \gamma^2 h' = Q(1+\epsilon\gamma^2 h') * [r<1]$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}rv - \gamma^2\Big(1+Q\epsilon * [r<1]\Big)h' = Q * [r<1]$$
$$\epsilon\frac{v^2}{r} + v = \frac{\partial}{\partial r}h'$$

with $\epsilon = |Q|/f_0$ the Rossby number and $\gamma^2 = f_0^2R^2/gH^2 = R^2/R_d^2$ the stratification parameter. Here $Q = \pm 1$.

- for cyclones, the velocities and thickness perturbations are weaker than predicted by QG theory. If, however, we simply drop the centrifugal term, the equations become linear. The deformation radii inside and outside are different: $R_d^{\mathrm{out}} = \sqrt{1+\epsilon}\ R_d^{\mathrm{in}}$. Note that the exact vorticity equation

$$\nabla^2\psi = h'q + Hq - f_0$$

with the geostrophic approx. becomes

$$\nabla^2\psi - \frac{f_0 q}{g}\psi = Hq - f_0 = Q$$

so that we are replacing a factor f_0/H by q in the definition of R_d. The velocities from this are very close, but the height field is a bit weaker since it does not have to compenstate the centrifugal terms.

- for anticyclones, the velocities and thickness perturbations are stronger than QG; again with the variable R_d, the velocities are similar, but the heights are now weaker, since v^2/r is opposite in sign to fv. The ϵ values are limited; when they become greater than 1, we have "anomalous highs" with a reversed central pressure.
- Steady flow: in this case

$$\mathbf{u} = \frac{1}{h}\hat{\mathbf{z}}\times\nabla\Psi$$

and

$$\mathbf{u}\cdot\nabla B = 0 \ \ , \ \ B = gh + \frac{1}{2}\left(\frac{|\nabla\Psi|}{h}\right)^2 = B(\Psi) \ \ , \ \ \mathbf{u}\cdot\nabla q = 0 \ \ , \ \ q = \frac{\nabla\cdot\frac{1}{h}\nabla\Psi + f}{h} = q(\Psi)$$

These are not independent:

$$\frac{\partial B}{\partial \Psi} = \frac{\frac{\partial}{\partial x}B}{\frac{\partial}{\partial x}\Psi} = \frac{(\zeta + f)v}{hv} = q$$

If we know $q(\Psi)$, we can integrate to get $B(\Psi)$, solve the cubic equation to get $h(\Psi)$ and then solve the PV equation to get Ψ. Note that steadily propagating solutions have

$$\mathbf{u} - c\hat{\mathbf{x}} = \hat{\mathbf{z}} \times \nabla\Psi \quad , \quad B(\Psi) = gh + \frac{1}{2}|\mathbf{u} - c\hat{\mathbf{x}}|^2 + c\int^y f$$

with the same q and relationship $\partial B/\partial\Psi = q$. But this inversion is not necessarily going to be unique: You could have a wavelike feature or a zonal flow. In QG, for example,

$$[\nabla^2 - R_d^{-2}]\psi + \beta y = -K^2(\psi - cy) \quad , \quad c = -\beta/K^2$$

has wave-like, Bessel function, zonal flow solutions, complex patterns,...

More critical, though, is the fact that this procedure is not the same as inverting $q(x, y)$. So, what can we do in that case?

- Represent Ψ as a polynomial in q with unknown coefficients.

$$\Psi = A_n q^n \quad \Rightarrow \quad B = A_n \frac{n}{n+1} q^{n+1} + B_0 = gh + \frac{1}{2h^2}[A_n n q^{n-1}]^2 |\nabla q|^2$$

  Solve this cubic for h. Put that in

$$qh = \nabla \frac{A_n n q^{n-1}}{h} \nabla q + f$$

  and minimize the error with respect to A_n and B_0.

- Let's consider a more general inversion using the Hemholtz decomposition and look at the horizontal divergence equation, dropping all the terms involving D (and therefor φ). We can start from the momentum equations with φ dropped

$$\frac{\partial}{\partial t}\hat{\mathbf{z}} \times \nabla\psi + -(\nabla^2\psi + f)\nabla\psi = -\nabla[gh + \frac{1}{2}|\nabla\psi|^2]$$

  taking minus the divergence gives a nonlinear balance equation

$$\nabla \cdot (\nabla^2\psi + f)\nabla\psi + \frac{1}{2}\nabla^2|\nabla\psi|^2 = \nabla^2 gh$$

  which can be combined with the PV

$$\nabla^2\psi + f = qh$$

  to solve for ψ and h. The PV evolves by the rotational flow

$$\frac{\partial}{\partial t}q + J(\psi, q) = 0$$

We could approach this iteratively: start with $h \equiv H + f_0\psi/g + h'$, so that the PV equation is

$$\nabla^2\psi - \gamma^2\psi = Q - \beta y + \left[Q\frac{f_0\psi}{gH} + Q\frac{h'}{H} + f_0\frac{h'}{H}\right]$$

and the divergence/ balance equation yields

$$\nabla^2 gh' = \nabla\cdot(\nabla^2\psi + \beta y)\nabla\psi - \frac{1}{2}\nabla^2|\nabla\psi|^2 = \nabla\cdot\beta y\nabla\psi + 2J(\psi_x, \psi_y)$$

We solve the PV equation for ψ, neglecting the terms in square brackets, then solve the balance eqn. for h'. We substitute these estimates into the square bracket terms and iterate.

- Note that for the radially symmetric case with $\beta = 0$, this procedure gives the correction term

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}gh' = \frac{2}{r}\frac{\partial(V\cos\theta, V\sin\theta)}{\partial(r,\theta)} = \frac{2}{r}V_r V = \frac{1}{r}\frac{\partial}{\partial r}V^2$$

implying

$$r\frac{\partial}{\partial r}gh' = V^2$$

- However, the procedure may not converge; for cyclones with Rossby number bigger than $\frac{1}{2}$ (at least for $\gamma = 1$) it diverges immediately. But it does converge with about 5 iterations for both cyclones and anticyclones with Rossby number less than about 0.2.

Balance equations - 1 page

From the SW equations $[\epsilon_{3ij} \to \epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\epsilon_{ij}\epsilon_{jk} = -\delta_{ik}]$

$$\frac{\partial}{\partial t} u_i - \epsilon_{ij}(f+\zeta)u_j = -\nabla_i \left[gh + \frac{1}{2} u_j u_j \right]$$
$$\frac{\partial}{\partial t} h + \nabla_i h u_i = 0$$

We derive the vorticity equation ($\zeta = \epsilon_{mi}\nabla_m u_i$)

$$\frac{\partial}{\partial t}\zeta + \nabla_m (f+\zeta) u_m = 0$$

or

$$\frac{\partial}{\partial t} hq + \nabla_m hq u_m = 0$$

With the mass equation, we then find conservation of PV

$$\frac{\partial}{\partial t} q + u_m \nabla_m q = 0 \tag{PV}$$

The divergence equation ($D = \nabla_i u_i$)

$$\frac{\partial}{\partial t} D - \epsilon_{ij}\nabla_i[(f+\zeta)u_j] = -\nabla^2 \left[gh + \frac{1}{2} u_j u_j \right]$$

We use this to derive a balance condition by neglecting the divergent flow so assuming that $u_i = -\epsilon_{ij}\nabla_j \psi$ so that $D \simeq 0$ and

$$\epsilon_{ij}\nabla_i[(f+\zeta)\epsilon_{jk}\nabla_k \psi] = -\nabla^2 \left[gh + \frac{1}{2}|\nabla\psi|^2 \right]$$

or

$$\nabla^2 \left[gh + \frac{1}{2}|\nabla\psi|^2 \right] = \nabla \cdot (f + \nabla^2\psi)\nabla\psi \tag{B}$$

Given the PV

$$q = \frac{\nabla^2\psi + f}{h} \tag{q}$$

we can solve (q) and (B) for ψ and h and then step the PV forward using (PV) in the form $\mathbf{u} = \hat{\mathbf{z}} \times \nabla\psi$ in

$$\frac{\partial}{\partial t} q + J(\psi, q) = 0$$

• Approximate inversion: if we split h into its mean, the geostophic part, and the rest

$$h = H + \frac{f_0}{g}\psi + h'$$

and write $Q = Hq - f_0$ or $q = (Q + f_0)/H$, the PV evolution is still conservation of Q, but

$$(\nabla^2 - \gamma^2)\psi = Q - \beta y + Q\left(\frac{f_0\psi}{gH} + \frac{h'}{H}\right) + f_0\frac{h'}{H} \qquad (q1)$$

$$\nabla^2 gh' = \nabla \cdot \beta y \nabla\psi + \nabla \cdot \nabla^2\psi\nabla\psi - \nabla^2\frac{1}{2}\nabla\psi|^2 = \nabla \cdot \beta y \nabla\psi + 2J(\psi_x, \psi_y) \qquad (B1)$$

Iteration: (1) given Q, take only the first two terms on the rhs of (q1), find ψ. (2) put this in (B1) and find h'. Put ψ anf h' in the rhs of (q1). (3) find an improved ψ. Repeat (2) and (3) until converged.

- QG: $h' = 0$ and drop $Q\psi$ terms. Scaling arguments.
- slightly modified QG: combine

$$\gamma^2 + \frac{f_0 Q}{gH} = \frac{f_0}{g}q$$

as coefficient of ψ.

• Omega eqn: we can estimate the divergence from the exact form

$$\frac{\partial}{\partial t}h + J(\psi, h) = \nabla \cdot h\nabla\varphi \simeq H\nabla^2\varphi$$

From inversions at two time steps, we can estimate $\frac{\partial}{\partial t}h$ and $J(\psi, h)$. This allows us to solve for φ. Note that his gives us the vertical velocity, since

$$w = \frac{z}{h}\frac{D}{Dt}h = -z\nabla \cdot \mathbf{u} = z\nabla^2\phi \quad \Rightarrow \quad w(H) = \frac{\partial}{\partial t}h + J(\psi, h)$$

In QG, this is even simpler, since $h \simeq f_0\psi/g$. Then

$$\frac{\partial}{\partial t}\psi \simeq \frac{g}{f_0}w$$

But $\frac{\partial}{\partial t}(\nabla^2 - \gamma^2)\psi = -J(\psi, Q)$ (with $Q = \nabla^2\psi - \gamma^2\psi + \beta y$) so that

$$(\nabla^2 - \gamma^2)w = \frac{f_0}{g}\frac{\partial}{\partial t}Q = -\frac{f_0}{g}J(\psi, Q)$$

• Piecewise inversion: in the lab, we do a 3D version, but the fundamental question and approach applies here as well: suppose we have two patches of PV and see the evolution of each. Are the changes in patch (1) associated mostly with the flow from the Q_1 anomalies or from Q_2? In QG, we can answer that precisely: from Q_1 we find $\psi_1 = \int G(\mathbf{x}, x')Q_1(\mathbf{x}')$ and likewise ψ_2. These give $\mathbf{u}_1$ and $\mathbf{u}_2$. We then have

$$\frac{\partial}{\partial t}Q_1 = -\mathbf{u}_1 \cdot \nabla Q_1 - \mathbf{u}_2 \cdot \nabla Q_1$$

and can assess the advection of Q_1 by $\mathbf{u}_1$ (self-interaction) and by $\mathbf{u}_2$ (forced by other). The sum of these two $\frac{\partial}{\partial t}Q_n$ equations gives exactly the $\frac{\partial}{\partial t}Q$ equation.

But the balance equations have quadratic terms, so solving for ψ_1, h_1' by setting $Q_2 = 0$ and for ψ_2, h_2' by taking $Q_1 = 0$ does not give $\psi = \psi + 1 + \psi_2$ – i.e., if we look at the $(f_0/gH)Q\psi$ term, it has $Q_1\psi_1 + Q_2\psi_2 + Q_1\psi_2 + Q_2\psi_1$ and the last two terms will not appear with the procedure above. The advection is ok, but the split inversion is not. Davis and Emanuel (1991) propose a resolution by splitting the equations as $Q\psi = \frac{1}{2}(Q\psi_1 + Q_1\psi) + \frac{1}{2}(Q\psi_2 + Q_2\psi)$ with now $\psi = \psi_1 + \psi_2$ and $Q = Q_1 + Q_2$. Then the equations [ignoring β for simplicity] become

$$\begin{aligned}(\nabla^2 - \gamma^2)\psi_1 &= Q_1\left[1 + \frac{f_0\psi}{2gH} + \frac{f_0h'}{2H}\right] + \frac{f_0Q}{2gH}\psi_1 + \frac{Q}{2H}h_1' + \frac{f_0}{H}h_1' \\ \nabla^2 gh_1' &= J(\psi_x, \psi_{1y}) + J(\psi_{1x}, \psi_y)\end{aligned}$$

Note: these equations are linear in ψ_1, h_1 but have variable coefficients. We first invert the full field nonlinearly to find ψ, h', and secondly solve the above two equations to find ψ_1, h_1' and likewise for ψ_2, h_2. We can then evaluate $\mathbf{u}_1 \cdot \nabla Q_1$ and $\mathbf{u}_2 \cdot \nabla Q_2$ to see the self vs. other interactions.

• Eddy Propagation: consider an isolated eddy with pressure anomaly gh' ($h = h_0 + h'$). The mass equation implies

$$\frac{\partial}{\partial t}V' \equiv \frac{\partial}{\partial t}\int h' = -\oint \mathbf{u} \cdot \hat{\mathbf{n}}h = 0$$

as long as velocities decay rapidly enough. Define $\langle S \rangle = \int S/V'$. Still from the mass equation

$$\frac{\partial}{\partial t}X_i = \langle u_i h \rangle \quad , \quad X_i = \langle x_i h' \rangle$$

The acceleration

$$\frac{\partial^2}{\partial t^2}X_i = \langle \frac{\partial}{\partial t}u_i h \rangle = -\langle \nabla_j u_i u_j h \rangle + \epsilon_{ij}\langle f u_j h \rangle + g\langle \nabla_i [h_0 h' + \frac{1}{2}h'^2] \rangle$$

from the momentum equations gives

$$\frac{\partial^2}{\partial t^2}X_i = \epsilon_{ij}(f_0 + \beta Y)\frac{\partial}{\partial t}X_j + \beta\epsilon_{ij}\langle (y - Y)u_j h \rangle$$

The second derivative term corresponds to wobbling of the vortex by inertial waves or rotation if elliptical. The translation comes from the beta term

$$\frac{\partial}{\partial t}X_i = -\frac{\beta}{f}\big\langle (y-Y)u_i h\big\rangle$$

For a steadily translating eddy

$$(u-c)h = -\frac{\partial}{\partial y}[\Psi + ch_0 y] \quad or \quad uh = \frac{\partial}{\partial y}\Psi + ch' \quad and \quad vh = \frac{\partial}{\partial x}[\Psi + ch_0 y]$$

giving

$$\big\langle (y-Y)uh\big\rangle = \big\langle \Psi \big\rangle \quad , \quad \big\langle (y-Y)vh\big\rangle = 0$$

so that

$$c = -\frac{\beta}{f}\big\langle \Psi \big\rangle$$

Note that this is westward for cyclones ($\Psi < 0$ and $h' < 0$) or anticyclones (both positive). If we use a geostrophic estimate

$$\mathbf{u}h \sim \hat{\mathbf{z}} \times \nabla\Psi \sim \frac{gh_0}{f}\hat{\mathbf{z}} \times \nabla[h' + \frac{h'^2}{2h_0}]$$

$$c \sim -\beta R_d^2[1 + \big\langle h'^2/h_0 \big\rangle]$$

The factor in brackets is bigger than one for anticyclones – they move faster than the longest linear wave, but is negative for cyclones.

Stratified flows

Quasi-geostrophic equations and pseudo-PV

We'll look at various derivations. All are based on $b = \int^z N^2(z')dz' + b'(\mathbf{x}, t)$, near geostrophic balance $\mathbf{u} \simeq \mathbf{u}_g = \frac{1}{f}\hat{\mathbf{z}} \times \nabla\phi'$, and hydrostatic balance $\frac{\partial}{\partial z}\phi' = b'$, and scales small compared to the size of the earth ($\beta y << f$)

• Simplest derivation from PV: use the hydrostatic boussinesq eqns in b coords

- the PV can be written

$$\tilde{q} = H(b)q - f_0 = \frac{1}{1 + h'/H}\left[\zeta + \beta y - \frac{f_0}{H}h'\right]$$

- proceed as in the SW equations to get the QGPV

$$Q = \nabla^2\psi + \beta y - \frac{f_0}{H}h' \quad , \quad \frac{\partial}{\partial t}Q + J(\psi, Q) = 0$$

- Use the vertical equation with $M = p/\rho_0 - bz$

$$\frac{\partial^2}{\partial b^2}M = -h'$$

and geostrophy to find

$$Q = \nabla^2\psi + \frac{f_0^2}{H(b)}\frac{\partial^2}{\partial b^2}\psi + \beta y$$

- return to z coords using

$$H \simeq 1/N^2 \quad and \quad \frac{\partial}{\partial b} \sim H\frac{\partial}{\partial z}$$

• From momentum: we'll use the anelastic equations (appendix) in Lagrangian derivative notation

$$\frac{\partial}{\partial t}\mathbf{u} + \boldsymbol{\zeta} \times \mathbf{u} + \frac{1}{2}\nabla|\mathbf{u}|^2 = \frac{D}{Dt}\mathbf{u}$$

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla\phi + b\hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{u} + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}w = 0$$

$$\frac{\partial}{\partial t}b = 0$$

with $\overline{\rho}(z)$ the density of an **isentropic** fluid.

- define the geostrophic and ageostrophic velocities

$$\mathbf{u}_g = \frac{1}{f}\hat{\mathbf{z}} \times \nabla\phi \quad , \quad \mathbf{u}_a = \mathbf{u} - \mathbf{u}_g$$

- then the horizontal equations become

$$\frac{D}{Dt}\mathbf{u} = -f\hat{\mathbf{z}} \times \mathbf{u}_a$$

The acceleration comes from the weak ageostrophic flows.

- approximate the acceleration by geostrophic advection of geostrophic momentum

$$\mathbf{u}_a = \hat{\mathbf{z}} \times \frac{1}{f}\frac{D_g}{Dt}\mathbf{u}_g$$

At this order, we can ignore β and use $\mathbf{u}_g = \hat{\mathbf{z}} \times \nabla\psi$ with $\psi = f_0\phi$. Therefore

$$\mathbf{u}_a = -\frac{1}{f_0}\frac{D_g}{Dt}\nabla\psi \quad , \quad \frac{D_g}{Dt} = \frac{\partial}{\partial t} + \psi_x\frac{\partial}{\partial y} - \psi_y\frac{\partial}{\partial x} = \frac{D_\psi}{Dt}$$

- Use this in the mass equation

$$\nabla \cdot \mathbf{u}_g + \nabla \cdot \mathbf{u}_a = -\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}w$$

and show that this corresponds to absolute vorticity changing from vortex stretching of f_0.

$$\frac{D_\psi}{Dt}(\nabla^2\psi + \beta y) = f_0\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}w \qquad (zeta)$$

- Using the hydrostatic equation and geostrophy to find $b' = f_0\psi_z$. Use this in the thermodyn eqn

$$\frac{D_\psi}{Dt}b' + wN^2 = 0 \qquad (b)$$

to find the rhs of the vorticity equation. Be careful with $\frac{\partial}{\partial z}(\mathbf{u}_g \cdot \nabla b')$

- You now have the PV equation

$$Q = \nabla^2\psi + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}\frac{\partial}{\partial z}\psi' + \beta y \quad , \quad \frac{\partial}{\partial t}Q + J(\psi, Q) = 0$$

with the added advantage that the boundary conditions are still simple $w = 0$ on a flat surface.

- the omega equation and secondary circulations
  - eliminate the $\frac{\partial}{\partial t}$ from (zeta) and (b) to find a diagnostic equation for w (ω in pressure coords).
  - TEM approach: take the buoyancy equation

$$\frac{\partial}{\partial t}b' + J(\psi, b') + N^2 w = 0$$

and define the transformed Eulerian w as

$$w^+ = w + J(\psi, b'/N^2)$$

- how can you define the TEM ageostrophic velocity to maintain the same value of $\nabla \cdot u_a + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}w$ with U_a replaced by U_a^+ and w by w^+?
- Note that the vorticity equation becomes

$$\frac{\partial}{\partial t}\zeta + J(\psi, Q) = f_0 \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}w^+$$

  Combine this with the modified thermo eqn

$$\frac{\partial}{\partial t}\psi_z + \frac{N^2}{f_0}w^+ = 0$$

  to find the PV equation and the equivalent to the omega equation.

• the superposition principle: just as in the BTVE ans QG SW system, we can solve for the flow from individual PV anomalies and sum them up.

$$Q = \beta y \sum Q_i \quad \Rightarrow \quad \psi = \sum \psi_i \quad with \quad (\nabla^2 + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial z})\psi_i = Q_i$$

- What do point PV vortices look like? Just take $\overline{\rho}$ and N^2 to be constant.
- For an isothermal atmosphere (yes, you need to go back either to the original equations or use the $p_\xi = -\overline{\rho}g$ coordinates)

$$\overline{\rho} = \rho_0 \exp(-z/H) \quad , \quad N^2 = \frac{\gamma - 1}{\gamma}\frac{g}{H} \quad , \quad H = RT/g$$

  and

$$\nabla^2 + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial z} \to \nabla^2 + \frac{f_0^2}{N^2}\left(\frac{\partial^2}{\partial z^2} - \frac{1}{H}\frac{\partial}{\partial z}\right)$$

  Consider the inversion equation in terms of

$$\psi = \psi' e^{z/2H}$$

  What do point vortices look like now?
- Boundary conditions: The operator to be inverted is a second order elliptic operator, so it requires boundary conditions on the surface bounding the 3D domain. Let's consider the bottom at $z = 0$. The boundary condition is

$$w = 0 \quad \Rightarrow \quad \frac{D_\psi}{Dt}b' = 0$$

  The surface temperature (buoyancy) is determined by the initial condition and advection. But

$$\psi_z(\mathbf{x}, 0, t) = \frac{1}{f_0}b'(\mathbf{x}, 0, t)$$

  so this provides a boundary condition.

- we can do a bunch of inversion exercizes now

1) Start with $\beta = 0$, $\overline{\rho} = \rho_0$, $N^2 = const$
   a) $b'(\mathbf{x}, 0) = A\cos(kx)$, $Q = 0$, infinite domain; pay particular attention to warm/cyclonic, cold/anticyclonic
   b) $b'(\mathbf{x}, 0) = A\cos(kx)$, $b'(\mathbf{x}, H) = 0$
   c) $b'(\mathbf{x}, 0) = 0$, $b'(\mathbf{x}, H) = A\cos(kx)$ warm? cold?
   d) $Q = A\cos(kx)\cos(\pi z/H)$, no boundary anomalies
   e) General
$$\psi = \psi_0 + \psi_I + \psi_H$$
   with

$(\nabla^2 + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial z})\psi$	$\psi_z(0)$	$\psi_z(H)$
$Q - \beta y$	0	0
0	$b'(0)/f_0$	0
0	0	$b'(H)/f_0$

2) Isothermal atmosphere
   a) $b'(\mathbf{x}, 0) = A\cos(kx)$, $Q = 0$
3) Standard atmosphere: the standard atmosphere is made up of piecewise linear T segments, with $R = 287.05$
4) Ocean profile

Appendix: Anelastic equations

We define a basic state which is hydrostatic

$$\nabla\overline{p} = -\overline{\rho}\nabla\Phi$$

and let $p = \overline{p} + \rho\phi$, $\rho = \overline{\rho}/(1+\delta)$. Then the pressure and geopotential terms become

$$\begin{aligned} -\frac{1}{\rho}\nabla p \; - \nabla\Phi &= (1+\delta)\nabla\Phi - \nabla\phi - \phi\nabla\ln\overline{\rho} + \phi\nabla\ln(1+\delta) - \nabla\Phi \\ &= -\nabla\phi + (\delta\nabla\Phi - \phi\nabla\ln\overline{\rho}) + \phi\nabla\ln(1+\delta) \end{aligned}$$

The last term will be dropped under $\delta << 1$. The momentum equations become

$$\frac{\partial}{\partial t}\mathbf{u} + (\boldsymbol{\zeta} + 2\Omega)\times\mathbf{u} = -\nabla\left[\phi + \frac{1}{2}|\mathbf{u}|^2\right] + b\,\hat{\mathbf{z}}$$

with the buoyancy being

$$b = \delta\nabla\Phi - \phi\nabla\ln\overline{\rho}$$

When we drop the $\frac{D}{Dt}\ln(1+\delta)$ term in the mass equation, it becomes

$$\nabla\cdot\overline{\rho}\mathbf{u} = 0$$

We define the coefficients of expansion

$$d\,\ln\rho = -\alpha d\theta + \beta dS + \gamma dp$$

We now take the basic state to be isentropic and constant S, so that

$$\nabla\ln\overline{\rho} = -\overline{\gamma}\,\overline{\rho}\,\nabla\Phi \quad , \quad b = g(\delta + \overline{\gamma}\,\overline{\rho}\,\phi)$$

Using

$$\begin{aligned} \ln(1+\delta) \simeq \delta &= \ln\overline{\rho} - \ln\rho(\overline{\theta}+\theta', \overline{S}+S', \overline{p}+\rho\phi) \\ &\simeq \overline{\alpha}\theta' - \overline{\beta}S' - \overline{\gamma}\,\overline{\rho}\,\phi \end{aligned}$$

gives

$$b = \overline{\alpha}g\theta' - \overline{\beta}gS'$$

For an ideal gas (no S and $\ln\theta = -\ln\rho + \frac{1}{\gamma}\ln p$)

$$b = g\frac{\theta'}{\overline{\theta}}$$

- Show this directly from

$$\ln\overline{\theta}(1+\theta'/\overline{\theta}) = -\ln\overline{\rho}/(1+\delta) + \frac{1}{\gamma}\ln(\overline{p}+\overline{\rho}\phi)$$

  (negligibly different – $p = \overline{p} + \overline{\rho}\phi$)
- find δ in terms of θ' and p'
- evaluate $RHS = -\frac{1}{\rho}\nabla p - \nabla\Phi$
- find $\frac{\partial}{\partial z}\ln\overline{\rho}$ when $\overline{\theta}$ is constant. Use $\nabla\overline{p} = -\overline{\rho}\nabla\Phi$.
- use this to find $RHS \simeq -\nabla\phi + g\theta'/\overline{\theta}$

Energy eqns

The KE equation is

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{\rho}|\mathbf{u}|^2 + \nabla\cdot\left[\overline{\rho}\mathbf{u}(\phi+\frac{1}{2}|\mathbf{u}|^2)\right] = g\overline{\rho}\,\overline{\alpha}\,w\theta' - g\overline{\rho}\,\overline{\beta}\,wS'$$

Given the profile of $\overline{\alpha}(z)$ and $\overline{\beta}(z)$, we define A and B as their integrals so that

$$\overline{\rho}\frac{D}{Dt}A\theta' = \overline{\rho}\,\overline{\alpha}\,w\theta'$$

So that

$$\frac{\partial}{\partial t}E + \nabla\cdot\mathbf{u}E + \nabla\cdot\overline{\rho}\mathbf{u}\phi = 0 \quad , \quad E = \frac{1}{2}\overline{\rho}|\mathbf{u}|^2 + \overline{\rho}gA\theta' - \overline{\rho}gBS'$$

- Note – isentropic atmospheres are finite. We can see in the simple case

$$\frac{\rho_0}{\rho}\left(\frac{p}{p_0}\right)^{1/\gamma} = \frac{\theta}{\theta_0} = 1$$

so the hydrostatic eqn. gives

$$gz = \frac{p_0}{\rho_0(1-1/\gamma)}\left[1-\left(\frac{p}{p_0}\right)^{1-1/\gamma}\right] \quad \Rightarrow \quad z(p=0) = \frac{H}{1-1/\gamma} \sim 3.5H \sim 30km$$

- Ertel PV

$$\frac{D}{Dt}q = 0 \quad , \quad q = \frac{(\nabla\times\mathbf{u} + f\hat{\mathbf{z}})\cdot\nabla b}{\overline{\rho}}$$

Appendix II: Not isentropic?

We will use the hydrostatic, pressure-like coordinates defined by

$$\frac{\partial}{\partial \xi} p = -g\overline{\rho}(\xi)$$

Following the notes (p4), but replacing ϕ by $\phi + g\xi$ gives

$$\frac{D}{Dt}\mathbf{u} + \mathbf{f} \times \mathbf{u} = -\nabla\phi$$

$$\frac{\partial}{\partial \xi}\phi = g\frac{\overline{\rho} - \rho}{\rho} = b$$

$$\nabla \cdot \overline{\rho}\mathbf{u} + \frac{\partial}{\partial \xi}\overline{\rho}\omega = 0$$

$$\frac{\partial}{\partial t}b + \mathbf{u} \cdot \nabla b + \omega\left(N^2 + \frac{\partial b}{\partial \xi} - b\frac{\partial}{\partial \xi}\ln\overline{\rho} - \frac{2gb + b^2}{c_s^2}\right) = 0$$

$$N^2 = -g\frac{\partial}{\partial \xi}\ln\overline{\rho} - \frac{g^2}{c_s^2}$$

The QG formulation will follow in the same way: N^2 (with c_s^2 determined from $\overline{\rho}$ and the associated p) will be the only term remaining in the vertical buoyancy advection. So we still have

$$\frac{\partial}{\partial t}Q + J(\psi, Q) = 0 \quad , \quad Q = \nabla^2\psi + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\psi + \beta y$$

But the boundary condition at the bottom is no longer $\omega = 0$, and the bottom is not at $\xi = 0$. Recall that $\omega = \frac{D}{Dt}\xi$ and that ξ surfaces are constant pressure surfaces. Therefore the ground would only be a constant ξ surface if the surface pressure were constant. Instead, the surface pressure, equivalently $\xi_s(x, y, t)$, is related to the dependent variables by

$$g\xi_s + \phi(x, y, \xi_s, t) = 0$$

and

$$\omega(x, y, \xi_s, t) = \frac{D}{Dt}\xi_s$$

Consistent with the small ω in QG, we can approximate

$$\xi_s \simeq -\frac{1}{g}\phi(x, y, 0, t) \quad , \quad \omega(x, y, 0, t) = -\frac{1}{g}\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\phi(x, y, 0, t)$$

The buoyancy equation at the ground becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\left(b - \frac{N^2}{g}\phi\right) = 0$$

and the boundary condition relates this conserved property to ψ

$$\frac{\partial}{\partial t}\tilde{\theta}_s + J(\psi, \tilde{\theta}_s) = 0 \quad , \quad \tilde{\theta}_s = \frac{\partial\psi}{\partial \xi} - \frac{N^2}{g}\psi$$

where we put a tilde over $\tilde{\theta}_s$ to remind us that it is not the potential temperature $\theta(\xi = 0)$ [whose variations are proportional to ψ_z], but rather the potential temperature at the actual ground ξ_s.

Appendix IIa: Isothermal atmosphere

For an ideal gas, it's even simpler: we can use the relationship between potential temperature and density to write the hydrostatic eqn as

$$\frac{\partial}{\partial \xi}\phi = g\frac{\overline{\rho}-\rho}{\rho} = g\frac{\theta-\overline{\theta}}{\overline{\theta}} \quad \Rightarrow \quad \theta = \overline{\theta}(1+b/g)$$

Conservation of entropy, which is proportional to $\ln\theta$. gives

$$\frac{1}{g}\frac{1}{1+b/g}\frac{D}{Dt}b + \omega N^2 = 0 \quad \Rightarrow \quad \frac{D}{Dt}b + \omega N^2(1+b/g) = 0$$

with $N^2 = g\frac{\partial}{\partial \xi}\ln\overline{\theta}$. In the QG approximation $b << g$, so we end up with the same buoyancy equation and QG equation as above. The boundary conditions work the same way

$$\frac{\partial}{\partial t}\tilde{\theta}_s + J(\psi, \tilde{\theta}_s) = 0 \quad , \quad \tilde{\theta}_s = \frac{\partial \psi}{\partial \xi} - \frac{N^2}{g}\psi$$

• For free waves, $\tilde{\theta}_s = 0$. Find the vertical structure and deformation radius for an isothermal atmosphere with $\overline{\rho} = \rho_0 \exp(-\xi/H)$. You should get $R_d = \sqrt{\gamma g H}/f$. For Earth's atmosphere, $\gamma = 1.4$, implying an equivalent depth of 11 km.

Ertel

1)

$$\frac{\partial}{\partial t}q = -q\frac{1}{\rho}\frac{\partial}{\partial t}\rho + \frac{1}{\rho}\frac{\partial \mathbf{Z}}{\partial t}\cdot\nabla\theta + \frac{1}{\rho}\mathbf{Z}\cdot\nabla\frac{\partial\theta}{\partial t}$$

2)

$$\frac{\partial}{\partial t}\mathbf{Z} + \nabla\times(\mathbf{Z}\times\mathbf{u}) = -\nabla\times\left[\frac{1}{\rho}\nabla p\right] + \nabla\times\mathbf{F} = \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nabla\times\mathbf{F}$$

$$\nabla_i\theta\epsilon_{ijk}\nabla_j A_k = \nabla_j[\nabla_i\theta\epsilon_{ijk}A_k] = \nabla_j[\epsilon_{jki}\nabla_i\theta A_k]$$

$$\nabla\theta\cdot\nabla\times(\mathbf{Z}\times\mathbf{u}) = \nabla\cdot[(\mathbf{Z}\times\mathbf{u})\times\nabla\theta] = \nabla\cdot[\mathbf{u}(\mathbf{Z}\cdot\nabla\theta) - \mathbf{Z}(\mathbf{u}\cdot\nabla\theta)] = \nabla\cdot(\rho\mathbf{u}q) - \mathbf{Z}\cdot\nabla(\mathbf{u}\cdot\nabla\theta)$$

$\Rightarrow$

$$\frac{\partial}{\partial t}q = -q\frac{1}{\rho}\frac{\partial}{\partial t}\rho - \frac{1}{\rho}\nabla\cdot(\rho\mathbf{u}q) + \frac{1}{\rho}\mathbf{Z}\cdot\nabla(\mathbf{u}\cdot\nabla\theta) + \frac{\nabla\theta\cdot(\nabla\rho\times\nabla p)}{\rho^3} + \frac{1}{\rho}\nabla\theta\cdot\nabla\times\mathbf{F} + \frac{1}{\rho}\mathbf{Z}\cdot\nabla\frac{\partial\theta}{\partial t}$$

3)

$$\frac{1}{\rho}\mathbf{Z}\cdot\nabla\frac{\partial\theta}{\partial t} = -\frac{1}{\rho}\mathbf{Z}\cdot\nabla(\mathbf{u}\cdot\nabla\theta) + \frac{1}{\rho}\mathbf{Z}\cdot\nabla H$$

$$\frac{\partial}{\partial t}q = -q\frac{1}{\rho}\frac{\partial}{\partial t}\rho - \mathbf{u}\cdot\nabla q - q\frac{1}{\rho}\nabla\cdot(\rho\mathbf{u}) + \frac{\nabla\theta\cdot(\nabla\rho\times\nabla p)}{\rho^3} + \frac{1}{\rho}\nabla\theta\cdot\nabla\times\mathbf{F} + \frac{1}{\rho}\mathbf{Z}\cdot\nabla H$$

4)

$$\frac{\partial}{\partial t}q = -\mathbf{u}\cdot\nabla q + \frac{\nabla\theta\cdot(\nabla\rho\times\nabla p)}{\rho^3} + \frac{1}{\rho}\nabla\theta\cdot\nabla\times\mathbf{F} + \frac{1}{\rho}\mathbf{Z}\cdot\nabla H$$

5)

$$\frac{D}{Dt}q = \frac{\nabla\theta\cdot(\nabla\rho\times\nabla p)}{\rho^3} + \frac{1}{\rho}\nabla\theta\cdot\nabla\times\mathbf{F} + \frac{1}{\rho}\mathbf{Z}\cdot\nabla H$$

6) If $\rho = \rho(\theta, p)$

$$\frac{D}{Dt}q = \frac{1}{\rho}\nabla\theta\cdot\nabla\times\mathbf{F} + \frac{1}{\rho}\mathbf{Z}\cdot\nabla H$$