Roadmap #3: Quasi-balanced Waves and Instabilities

QG Rossby Waves

$$\frac{\partial}{\partial t}Q + J(\psi, Q) = 0 \quad , \quad Q = \nabla^2\psi + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\psi + \beta y \equiv \mathcal{L}\psi + \beta y$$

$$\frac{\partial}{\partial t}\theta + J(\psi, \theta) = 0 \quad , \quad \theta = \frac{\partial}{\partial z}\psi \quad , \quad z = 0, H$$

Specified background state

$$Q = \mathcal{Q}(y, z) + Q'(x, y, z, t) \quad , \quad \psi = \Psi(y, z) + \psi'(x, y, z, t)$$

with

$$\mathcal{L}\Psi = \mathcal{Q} - \beta y \quad , \quad \mathcal{L}\psi' = Q'$$

In general, the [zonal] mean of Q' will not be zero. In the atmosphere (and often in the ocean), we consider zonal means

$$Q = \overline{Q}(y, z, t) + Q'(x, y, z, t) \quad , \quad \psi = \overline{\psi}(y, z, t) + \psi'(x, y, z, t)$$

with

$$\mathcal{L}\overline{\psi} = \overline{Q} - \beta y \quad , \quad \mathcal{L}\psi' = Q'$$

but now $\overline{Q'} = 0$, and, in general, $\overline{Q}$ will change with time.

- Derive the dynamical equations for $\overline{Q}$ and Q' by taking a mean of the PV equation and also looking at the residual.
- Show that the linearized equation for Q' is the same if we use either the $\mathcal{Q}$ or $\overline{Q}$ form, but the nonlinear equation is not.

• Interior PV waves: take $\mathcal{Q} = \beta y$, $Q' = -K^2\psi'$, $\theta_0 = 0$, $\theta_H = 0$
  - find the dispersion relation
  - show that this is an exact solution
  - find K^2 for waves $\exp(\imath\mathbf{k}\cdot\mathbf{x} - \imath\omega t)F(z)$ where

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}F = -R_d^{-2}F \quad , \quad F_z(0) = 0 \quad , \quad F_z(H) = 0$$

  - For N^2 and $\overline{\rho}$ constant (which will denote as the CONST case), find the eigenfunctions and eigenvalues F and R_d.
  - the isothermal atmosphere $N^2 = (\gamma - 1)g/\gamma H$, $\overline{\rho} = \rho_0\exp(-z/H)$, find F and R_d.

• Edge waves
  - Again start with the CONST case for a fluid extending from $z = 0$ to $z = \infty$. Let $\Psi = -syz$ and let $\beta = 0$. Find Θ. Write the dynamical equation for θ'.
  - Find ψ' assuming $\theta' = A\cos(\mathbf{k}\cot\mathbf{x} - \omega t)$.
  - Show the nonlinear terms again vanish
  - Find ω.

- What happens for an infinitely deep fluid with a top surface at $z = 0$?

- Topographically forced waves and vertical propagation
  - with topography $h(x, y)$, the bottom condition becomes $w = \frac{D_g}{Dt}h$. Show that

$$\left(\frac{\partial}{\partial t} + \psi_x \frac{\partial}{\partial y} - \psi_y \frac{\partial}{\partial x}\right)\left(\psi_z + \frac{N^2}{f}h\right) = 0$$

  - Consider $h = h_0 \cos(\mathbf{k} \cdot \mathbf{x})$, $\Psi = -Uy$, $\mathcal{Q} = \beta y$ (show). Find the flow for the CONST case. Note trapped vs propagating modes. For the latter, having a structure

$$\exp(\imath kx + \imath mz)$$

    find the proper m value using upward group velocity or energy flux.
  - for the isothermal case.
- Basin and global modes

Interacting QG Rossby waves: Baroclinic Instability

- Eady model
  - QG: CONST case with $\Psi = -syz$, $Q' = 0$
    1) For perturbations on the two boundaries like $\Re\theta(t)\exp(\imath\mathbf{k} \cdot \mathbf{x})$

$$\psi = \theta_0 F_0(z) + \theta_H F_H(z) \quad , \quad \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\bar{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}F = k^2 F$$

    with

$$\frac{\partial}{\partial z}F_0(0) = 1, \frac{\partial}{\partial z}F_0(H) = 0 \quad , \quad \frac{\partial}{\partial z}F_H(0) = 0, \frac{\partial}{\partial z}F_H(H) = 1$$

    2) Write the evolution equations for θ_0 and θ_H.
    3) There's a formal solution to

$$\frac{\partial}{\partial t}\mathbf{v} = -\imath k\mathbf{M}\mathbf{v} \quad \Rightarrow \quad \mathbf{v} = \mathrm{expm}(-\imath kt\mathbf{M})\mathbf{v}(0)$$

    The matrix exponential is defined by a Taylor series and can be calculated by using $\mathbf{M} = \mathbf{Z}\Omega\mathbf{Z}^{-1}$ with $\mathbf{Z}$ the eigenvectors and Ω a diagonal matrix of eigenvalues. We can use this directly

$$\tilde{\mathbf{v}} = \mathbf{Z}^{-1}\mathbf{v} \quad \Rightarrow \quad \frac{\partial}{\partial t}\tilde{\mathbf{v}} = -\imath\mathbf{k}\Omega\tilde{\mathbf{v}} \quad \Rightarrow \quad \tilde{v}_i = \tilde{v}_i(0)\exp(-\imath k\Omega_{ii}t)$$

    Instability will occur when one of the eigenvalues has a positive imaginary part.
    4) Take $\theta = \tilde{\theta}\exp(-\imath kct)$ so that

$$M_{ij}\tilde{\theta}_j = c\,\tilde{\theta}_i$$

    5) Find the conditions for $\Im(c) > 0$ or $\Im(\omega) > 0$.

- Balance and the Eady model We consider the 2D anelastic model with a uniform gradient of θ and pressure in the y-direction, $\alpha g\theta = N^2 z - fsy + b'(x,z,t)$, $\phi = \frac{1}{2}N^2z^2 - fsyz + f\psi(x,z,t)$, but no other variations (and no β)

$$\frac{D}{Dt}u - fv = -f\psi_x$$
$$\frac{D}{Dt}v = -fu + fsz$$
$$\frac{D}{Dt}w = -f\psi_z + b$$
$$\frac{\partial}{\partial x}\overline{\rho}u + \frac{\partial}{\partial z}\overline{\rho}w = 0$$
$$\frac{D}{Dt}b' - fvs + N^2 w = 0$$

The vorticity equation is

$$\frac{D}{Dt}\zeta = fv_z - b'_x \quad , \quad \zeta = u_z - w_x = \nabla \cdot \frac{1}{\overline{\rho}}\nabla\varphi$$

and

$$\frac{D}{Dt}v = -\frac{f}{\overline{\rho}}\varphi_z + fsz$$
$$\frac{D}{Dt}b' = fvs + \frac{N^2}{\overline{\rho}}\varphi_x$$
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{1}{\overline{\rho}}\varphi_z\frac{\partial}{\partial x} - \frac{1}{\overline{\rho}}\varphi_z\frac{\partial}{\partial z}$$

1) Consider the balance approx with $\varphi = s\int^z z\overline{\rho} + \varphi'$

$$fv_z = b'_x \quad \Rightarrow \quad v' = \psi_x \ , \quad b' = f\psi_z$$

and

$$\frac{D}{Dt}\psi_x = -\frac{f}{\overline{\rho}}\varphi'_z$$
$$\frac{D}{Dt}\psi_z - s\psi_x = \frac{N^2}{f\overline{\rho}}\varphi'_z$$
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + sz\frac{\partial}{\partial x} + \frac{1}{\overline{\rho}}\varphi'_z\frac{\partial}{\partial x} - \frac{1}{\overline{\rho}}\varphi'_x\frac{\partial}{\partial z}$$

Eliminating the $\frac{\partial}{\partial t}$ term gives a diagnostic equation for ϕ' while eliminating the terms on the rhs gives a prognostic equation for ψ.

2) For the linearized problem

$$\frac{D}{Dt} \to \frac{\partial}{\partial t} + sz\frac{\partial}{\partial x}$$

show that the prognostic equation is just the QG PV equation (linearized). The diagnostic equation for ψ is like the omega equation.

3) Unlike the QG approximation, the linear solution is not a solution to the nonlinear problem.

• Equilibration: For the QG model, the growing waves are an exact solution; the amplitude will not stabilize. However, they can be unstable and the secondary instability can halt the growth by feeding energy to neutral or stable waves (in the presence of friction). We'll return to this later as it may be important for oceanic BCI. We may also be able to stabilize the waves by divergent heat fluxes, either $\frac{\partial}{\partial y}\overline{v'b'}$ or $\frac{\partial}{\partial z}\overline{w'b'}$. These feed back on the **mean** buoyancy field and, by thermal wind, the **mean** zonal velocity field. Here we will adopt the zonal mean which is relevant for the atmosphere but more questionable in the ocean and assume $\overline{\rho}$ is constant.

$$\frac{\partial}{\partial t}\overline{b} = -\frac{\partial}{\partial y}[\overline{v}\overline{b} + \overline{v'b'}] - \frac{\partial}{\partial z}[\overline{w}\overline{b} + \overline{w'b'}]$$

Within the QG form, $\overline{v} = \overline{w} = 0$ and the $\overline{w'b'}$ term is also negligible. So we can change the mean density by divergence or convergence of the northward eddy heat flux. In the 2D equations $\frac{\partial}{\partial y} = 0$ and $\overline{w} = 0$ so that the only active term is the divergence of the vertical heat fluxes.

• Charney model

If we start from the isothermal, hydrostatic atmosphere with a basic state

$$u = U(\xi) \quad , \quad U(0) = 0; \quad , \quad \phi = -U(\xi)\int^y f \quad , \quad b = -U_z\int^y f$$

with the boundary condition

$$g\xi_s - U(\xi_s)\int^y f = 0 \quad \Rightarrow \quad \xi_s = 0$$

The linearized equations (using z instead of ξ and w instead of ω) become

$$\frac{\partial}{\partial t}u + U\frac{\partial}{\partial x}u + wU_z - fv = -\frac{\partial}{\partial x}\phi$$

$$\frac{\partial}{\partial t}v + U\frac{\partial}{\partial x}v + fu = -\frac{\partial}{\partial y}\phi$$

$$\Rightarrow$$

$$\frac{\partial}{\partial t}\zeta + U\frac{\partial}{\partial x}\zeta + \beta v - w_y U_z = f\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}w$$

$$\frac{\partial}{\partial z}\phi = b$$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}w = 0$$

$$\frac{\partial}{\partial t}b + U\frac{\partial}{\partial x}b - fU_z v + w(N^2 - N^2U_z\int^y f/g - U_{zz}\int^y f) = 0$$

boundary condition :

$$g\xi_s' - U_z(0)\xi_s'\int^y f + \phi(\mathbf{x}, 0, t) = 0 \quad \& \quad w = \frac{\partial}{\partial t}\xi_s$$

$$\Rightarrow$$

$$\frac{\partial}{\partial t}\left[\phi_z - \frac{N^2}{g}\phi - \frac{U_{zz}(0)\int^y f}{g - U_z(0)\int^y f}\phi\right] - U_z(0)fv = 0$$

The QG form drops the $-w_y s$ term in the vorticity, replaces f by f_0, has $b = f_0 \frac{\partial}{\partial z}\psi$, and drops the $U_z \int^y f$ term compared to g (or in Joe's analysis drops the entire term) and $U_{zz} \int^y f$ compared to N^2. These are consistent with $Ro = U_z H/f_0$ and $\beta L/f_0$ being small. Then, eliminating the vertical velocity gives the QGPV eqn

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) Q + \left(\beta - \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}U\right)\psi_x = 0$$

$$Q = \nabla^2\psi + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\psi$$

$$\frac{\partial}{\partial t}\left(\psi_z - \frac{N^2}{g}\psi\right) - U_z\psi_x = 0$$

For $U = sz$ and an isothermal stratification, the PV gradient term is just $\tilde{\beta} = \beta + \frac{s f_0^2}{N^2 H}$ and is constant so that

$$\left(\frac{\partial}{\partial t} + sz\frac{\partial}{\partial x}\right) Q + \tilde{\beta}\psi_x = 0 \quad , \quad Q = \nabla^2\psi + \frac{f_0^2}{N^2}\left(\frac{\partial^2}{\partial z^2} - \frac{1}{H}\frac{\partial}{\partial z}\right)\psi$$

$$\frac{\partial}{\partial t}\left(\psi_z - \frac{N^2}{g}\psi\right) - s\psi_x = 0$$

for solutions of the form $\exp(\imath k[x - ct] + \imath \ell y)$

$$(sz - c)Q + \tilde{\beta}\psi = 0 \quad , \quad Q = \mathcal{L}\psi - K^2\psi$$

$$c(\psi_z - \frac{N^2}{g}\psi) + s\psi = 0$$

- Neutral solutions: These have $c = 0$ and are of the form $P_n(z/H)e^{-az/H}$ with $a^2 + a - K^2N^2H^2/f^2 = 0$. The gravest mode is $P_1 = z/H$ and has $\beta_{nd} = \tilde{\beta}N^2H^2/f^2 sH = 1 + 2a$. The second is a quadratic plus a linear term with $\beta_{nd} = 2 + 4a$.
- Charney found that slightly shorter waves were unstable – they have growth rates order $\sqrt{K - K_n}$ (nondimensional $K's$); Burger argued that all the waves except for the neutral ones with $K = \frac{1}{2}\sqrt{(\frac{b}{n})^2 - 1}$ are unstable; Miles showed that the longer waves have growth rates order $[K_n - K]^{3/2}$. The equation can be transformed into the hypergeometric eqn.
- Numerics confirm the picture of high growth rates for the gravest mode short waves, decreasing as K increases. The growth rates for the long waves indeed remain small.

• Charney-Stern Theorem: Consider the general zonal flow instability problem with $U = U(y, z)$. The perturbation PV equation is

$$\frac{\partial}{\partial t}Q' + U\frac{\partial}{\partial x}Q' + \mathcal{Q}_y\frac{\partial}{\partial x}\psi = 0 \quad , \quad \mathcal{Q}_y = \beta - U_{yy} - \mathcal{L}U$$

and the boundary conditions (using the simpler anelastic form) are

$$\frac{\partial}{\partial t}b' + U\frac{\partial}{\partial x}b' + \mathcal{B}_y\psi'_x = 0 \quad @\ z = 0, H \quad with \quad b' = f_0\psi'_z \quad , \quad \mathcal{B}_y = -fU_z$$

We can rewrite these in terms of displacements of PV contours and theta contours on the boundaries

$$Q' = -\eta\mathcal{Q}_y \quad , \quad b'(0) = -\eta_0\mathcal{B}_y(0) \quad , \quad b'(H) = -\eta_H\mathcal{B}_y(H)$$

- Show that for all three

$$\frac{\partial}{\partial t}\eta + U\frac{\partial}{\partial x}\eta = v' = \psi'_x$$

- From this, show that

$$\frac{\partial}{\partial t}\frac{1}{2}\mathcal{Q}_y\overline{\eta^2} = -\overline{v'Q'}$$

  and

$$\frac{\partial}{\partial t}\frac{1}{2}\mathcal{B}_y(0)\,\overline{\eta_0^2} = -\overline{v'b'}$$

  with the overbar a zonal average. The growth of the waves depends in downgradient fluxes of PV or heat on the boundaries.
- Now consider the integrated PV flux. Use

$$Q' = v'_x - u'_y + \frac{f_0}{\overline{\rho}}\frac{\partial}{\partial z}\frac{\overline{\rho}}{N^2}b'$$

  Show that

$$\overline{\rho}\overline{v'Q'} = \nabla\cdot\left(-\overline{\rho}\overline{u'v'}\,\hat{\mathbf{y}} + \frac{f_0\overline{\rho}}{N^2}\overline{v'b'}\,\hat{\mathbf{z}}\right)$$

  the divergence of the Eliassen-Palm flux. For the PV flux integrate over y and z, therefore

$$\begin{aligned}\iint\overline{\rho}\overline{v'Q'} &= \int dy\left[\frac{f_0\overline{\rho}(H)}{N(H)^2}\overline{v'b'}|_H - \frac{f_0\overline{\rho}(0)}{N(0)^2}\overline{v'b'}|_0\right] \\ &= \frac{1}{2}\frac{\partial}{\partial t}\int dy\left[-\frac{f_0\overline{\rho}(H)}{N(H)^2}\mathcal{B}_y(H)\overline{\eta_H^2} + \frac{f_0\overline{\rho}(0)}{N(0)^2}\mathcal{B}_y(0)\overline{\eta_0^2}\right]\end{aligned}$$

- Combine this with the PV contours eqn to get

$$\frac{\partial}{\partial t}\frac{1}{2}\int dy\left[\int dz\left(\overline{\rho}\mathcal{Q}_y\overline{\eta^2}\right)-\frac{f_0\overline{\rho}(H)}{N(H)^2}\mathcal{B}_y(H)\overline{\eta_H^2}+\frac{f_0\overline{\rho}(0)}{N(0)^2}\mathcal{B}_y(0)\overline{\eta_0^2}\right]$$

$$=0$$

Therefore if

$$\mathcal{Q}_y(y,z)\geq 0 \quad and \quad \mathcal{B}_y(y,H)\leq 0 \quad and \quad \mathcal{B}_0(y,0)\geq 0$$

everywhere, the flow is stable. Likewise if the opposite signs hold for all three.
- Think about barotropic instability, Eady, Charney.
- Consider this result in terms of PV sheets. Replace the boundary conditions $\psi_z = f_0 b$ with $\psi_z = 0$ and take

$$\tilde{Q}=Q+\frac{f_0 b_0}{N^2(0)}\delta(z-0^+)-\frac{f_0 b_H}{N^2(H)}\delta(z-H^-)$$

Show that the integral of $\overline{\rho}\overline{v'Q'}$ now vanishes and that the condition on the displacements is just

$$\frac{\partial}{\partial t}\frac{1}{2}\iint\overline{\rho}\tilde{\mathcal{Q}}_y|\eta|^2=0$$

and we can satisfy the necessary condition for stability using with the interior gradients of the PV sheets or the PV sheets

$$\tilde{\mathcal{Q}}_y=\mathcal{Q}_y-\frac{f_0}{N^2}\mathcal{B}_y\delta(z-H^-)+\frac{f_0}{N^2}\mathcal{B}_y\delta(z-0^+)$$

must change sign.

- Critical layers: For normal-mode form

$$\eta=\frac{\psi'(y,z)}{U(y,z)-c}\exp(\imath k(x-ct))+c.c.$$

- show

$$\overline{\eta^2}=2\frac{|\psi'(y,z)|^2}{|U(y,z)-c|^2}\exp(2kc_i t)$$

and

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{\eta^2}=\frac{2kc_i|\psi'(y,z)|^2}{|U(y,z)-c|^2}\exp(2kc_i t)$$

For a growing disturbance $c_i > 0$, this is everywhere non-negative: the particles are getting further and further away from their position in the undisturbed flow. Therefore $\tilde{\mathcal{Q}}_y$ must have positive and negative regions.

- For neutral waves with c real, suppose there is a line $z_c(y)$ where $U(y, z_c) = c$. In a frame of reference moving with this speed

$$\frac{\partial}{\partial t}\eta + (U - c)\frac{\partial}{\partial x'}\eta = \frac{\partial}{\partial t}\eta = v(y, z_c)\exp(\imath k x')$$

so that the displacements just grow linearly if $v \neq 0$. If $\tilde{\mathcal{Q}}_y$ is non-zero, this implies there will be a PV fluz at the critical layer which cannot be balanced elsewhere since $\overline{\eta^2}$ is bounded everywhere else. We cannot have a neutal mode with a critical layer unless $\mathcal{Q}_y$ vanishes there or takes on different signs for different y's (recalling that z_c is a function of y; this only applies when U has both vertical and horizontal shear).
- In the limit as $c_i \to 0$, we can use $U - c_r \sim U_z(z_c)(z - z_c)$ and integrate across z_c to find

$$\frac{\partial}{\partial t}\frac{1}{2}\int_{z_c^-}^{z_c^+} \overline{\eta^2} = \frac{2k\pi|\psi'(y, z_c)|^2}{|U_z(y, z_c)|}$$

The related downgradient flux of PV must be balanced by growth of $\overline{\eta^2}$ elsewhere where the PV gradient is opposite to that in the critical layer.
- We avoided the critical layer issue previously by considering piecewise-constant $\mathcal{Q}$ fields in the shear layer and the Eady problem. For these $\mathcal{Q}_y$ is zero nearly everywhere.

Non-modal growth and superposition

• Consider growth of stable disturbances by superposition. Consider $U = sz$ in the CONST case but unbounded vertically. The perturbations will have PV anomalies

$$Q' = A\cos(kx + mz)$$

Show that

$$Q' = A\cos(k[x - szt] + mz) = A\cos(kx + [m - kst]z)$$

- Now invert to find ψ. Evaluate the KE $\langle K^2|\psi|^2\rangle$ and APE $\langle(f_0^2/N^2)|\psi_z|^2$ and the total. Show that it can grow significantly as the shear makes the vorticity barotropic and then decays again. This is the "Orr mechanism.' 'Look for large time and possibly large x solutions.

• In general if we have a perturbation equation

$$\frac{\partial}{\partial t}Q' + U\frac{\partial}{\partial x}Q' + \psi'_x\mathcal{Q}_y$$

and we deal with perturbations of the form $Q\exp(\imath kx)$ and use the Green's function to find ψ' in terms of Q'

$$\frac{\partial}{\partial t}Q' + \imath k\int dydz\,[U\delta(y - y')\delta(z - z') + \mathcal{Q}(y, z)G(y, z|y', z')]\,Q'(y', z')$$

$$= 0$$

If we discretize the Q' field at points y_i, z_i and use a discrete form of the integral, we have

$$\frac{\partial}{\partial t}Q'_i + \imath k M_{ij} Q'_j = 0$$

For the normal modes, $Q' \propto \exp(-\imath kct)$ we can see that the c values are the eigenvalues of $\mathbf{M}$.

- The general solution is

  $$Q'_i(t) = \text{expm}(-\imath kt M_{ij}) Q'_j(0)$$

  where this is a *matrix exponential* (`expm`). Show that this works in terms of the Taylor series expansion of $Q_i(t)$.
- use the decomposition in terms of the eigenfunctions and eigenvalues

  $$\mathbf{MZ} = \mathbf{Z}\Omega \quad \Rightarrow \quad \mathbf{M} = \mathbf{Z}\Omega\mathbf{Z}^{-1}$$

  where Ω is a diagonal matrix of the eigenvalues and the columns of $\mathbf{Z}$ are the associated eigenvectors. This gives another representation of the matrix exponential

  $$\text{expm}(-\imath kt\mathbf{M}) = \mathbf{Z}\exp(-\imath kt\Omega)\mathbf{Z}^{-1}$$

  with the exponential of a diagonal matrix just being an ordinary Matlab/Octave exponential of each element – in this case, just the diagonal elements. If any of the eigenvalues have positive imaginary parts, they will rapidly dominate $\exp(-\imath kt\Omega)$. But even if all the eigenvalues are real, we can get growth when they are not widely separated so that we can start with modes which are out of phase and amplify as they superimpose.
- We can find the amplification in terms of the *matrix norm*

  $$max\left(|Q'(t)|/|Q'(0)|\right) = ||\text{expm}(-\imath kt\mathbf{M})||$$

  This shows that for the Eady problem in the neutral regime, we can get amplifications of factors of 10 when close to the critical value, 2 or 3 further away.

• Another metric is the initial rate of amplification: what is the fastest a perturbation can grow? You might think it's the growth rate (or maximum growth rate if more than one mode is growing), but it's not.

- Why? Normal modes have to match the phase and amplitudes to keep the structure coherent; but we can find initial structures which maximize amplification at $t = 0$ but will not remain coherent.
- To make it quantitative, consider the "reactivity" (Neubert and Caswwll, 1997)

  $$\sigma_r = max\left[\frac{1}{|Q|}\frac{\partial}{\partial t}|Q|\right]$$

Using $|Q'|^2 = Q^H Q$ where Q^H is the transpose of the complex conjugate ($'$ in Matlab/Octave), show that this becomes

$$\sigma_r = max \frac{Q^H \frac{1}{2}[-\imath k M + \imath k M^H] Q}{Q^H Q}$$

This is given by the largest eigenvalue of $\frac{1}{2}[-\imath k M + \imath k M^H]$.

- show this is true in the generalized Eady problem by doing the optimazion directly holding $|Q| = 1$.
- computing the reactivity shows that it can be several times the normal mode growth rate and much more as the latter approaches zero.

Initial value problem

In wave problems, we tend to think of expanding in the normal modes, letting each evolve, and then reconstituting the results. But if we consider the Eady (CONST) problem as an example and want to consider the evolution of $\psi_0(x,z)$ or even $\psi_0(z)\exp(\imath k x)$, problems arise. For the dynamical equation

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right) Q' = 0 \quad , \quad Q' = \left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi = 0$$

the only normal modes are the two corresponding to growing or decaying waves or to the neutral waves with phase speeds on either side of the mean of the shear flow depending on the value of k. Clearly, these are not a complete set. Equally obviously, the initial disturbance, unlike the normal modes, does not have $Q' = 0$. We can solve the general initial value problem using Laplace transforms (c.f. Case, 1960).

• We'll consider an example with a potential interior reversal of PV gradient but $\psi_z = 0$ boundary conditions and $\exp(\imath k x)$ perturbations

$$\frac{\partial}{\partial t} Q' + \imath k U Q' + \imath k \mathcal{Q}_y \psi' = 0 \quad , \quad Q' = \frac{\partial^2}{\partial z^2}\psi - k^2 \psi$$

- Take the Laplace transform $q(s) = \int_0^\infty Q' \exp(-st)$

$$sq + \imath k U q + \imath k \mathcal{Q}_y p = Q'_0(z) \quad , \quad q = \frac{\partial^2}{\partial z^2} p - k^2 p$$

so that

$$\left[\frac{\partial^2}{\partial z^2} - k^2 + \frac{\imath k \mathcal{Q}_y}{s + \imath k U}\right] p = \frac{Q'_0(z)}{s + \imath k U}$$

- Write the solution using the Green's function

$$p = \int dz' G(z, z'|s) \frac{Q'_0(z')}{s + \imath k U(z')}$$

and express G in terms of two solutions to the homogeneous equation

$$G(z, z'|s) = \frac{1}{W(p_0, p_1|s)} p_0(z_<|s) p_1(z_>|s)$$

$$z_< = min(z, z') \quad , \quad z_> = max(z, z')$$

with

$$\left[\frac{\partial^2}{\partial z^2} - k^2 + \frac{\imath k \mathcal{Q}_y}{s + \imath k U}\right] p_n = 0 \quad , \quad \frac{\partial}{\partial z} p_0(0) = 0 \quad , \quad \frac{\partial}{\partial z} p_1(H) = 0$$

with the Wronskian being

$$W(p_0, p_1|s) = p_0 \frac{\partial p_1}{\partial z} - p_1 \frac{\partial p_0}{\partial z}$$

- Prove that W is independent of z.
- For the inversion

$$\psi(z, t) = \frac{1}{2\pi\imath} \int_{-\infty+a}^{\infty+a} ds\, e^{st} p(z|s)$$

with the contour taken to the right of the singularities. We have to close the contour to the left for $t > 0$, so will get contributions from all the poles or branch points. What are the singularities?

$$\psi(z, t) = \frac{1}{2\pi\imath} \int dz' Q_0'(z') \int_{-\infty+a}^{\infty+a} ds\, e^{st} \frac{G(z, z'|s)}{s + \imath k U(z')}$$

1) values of s where the Wronskian is zero. At these points, p_0 and p_1 are the same solution, satisfying both boundary conditions. These are the normal modes with $s = -\imath kc = -\imath kc_r + kc_i$. Unstable modes have poles in the right half-plane and their residues will contribute exponentially growing modes.
2) points on the imaginary axis where $s = -\imath kU(z_s)$ but $\mathcal{Q}_y(z_s) \neq 0$. Here p_0 and p_1 are singular. But the singularity is order $(z - \imath s/k)\ln(z - \imath s/k)$ with the p's also having with a constant term at the singular point.
3) The point $s = -\imath kU(z_0)$. Here we have a logarithmic singularity and a simple pole (from the constant term). The former gives $\psi \sim 1/t$ while the latter gives oscillations corresponding to advection by the flow at the singular point $\exp(\imath kU(z')t)$. Integrating these over z' will, as in the case considered above, give a signal decaying as $1/t$.
4) For the Eady problem, the singular modes satisfy

$$Q'(z, t) = Q'(z, 0) \exp(\imath kzt)$$

Solving this (computer algebra!) for $Q' = sin(\pi z)$ shows that $Q' \sim 1/t$ for large t; the initial transients die out slowly. But they can grow substantially for short times by the Orr mechanism above.

Spatial-temporal growth

Although almost all stability computations are for zonal flow and normal modes or at best wavenumber k, yet most atmospheric and oceanic disturbances are much more localized in space and time.

If we consider the evolution of a localized initial disturbance in 1D

$$\psi(x,t) = \int dk A(k) \exp(\imath kx - \imath\Omega(k)t)$$

(ignoring the issues of different modes and singular modes). We look for large time and possibly large x solutions

$$\psi(x,t) = \lim_{t\to\infty} \int dk A(k) \exp(\imath[kU - \Omega(k)]t)$$

with $U = x/t$.

- Deform the integration contour in the complex k plane to pass through the saddle point k_s such that $U = d\Omega/dk$. Then

$$\psi(x,t) \to A(k_s) \exp\left(\imath[k_s U - \Omega(k_s)]t\right) \int dk' \exp\left(-\imath t \frac{d^2\Omega}{dk^2}\bigg|_{k_s} k'^2\right)$$

  The envelope will be growing at a rate

$$\sigma = \Im[\Omega(k_s) - Uk_s]$$

- We can define the gradients of $\Omega - Uk$ as follows: suppose we specify the system in the frame moving at speed U as

$$\frac{\partial}{\partial t}\psi_i = -\imath k M_{ij}\psi_j$$

  Then ω is the eigenvalue of M with the largest imaginary part. Let its eigenvector be $\mathbf{z}$.

$$\mathbf{M}(k,U)\mathbf{z} = \omega\mathbf{z} \quad \Rightarrow \quad \mathbf{M}_k\mathbf{z} + \mathbf{M}\mathbf{z}_k = \omega_k\mathbf{z} + \omega\mathbf{z}_k$$

  (subcripts meaing derivatives) From the left eigenfunction $\mathbf{z}^+$ satisfying $\mathbf{z}^+\mathbf{M} = \omega\mathbf{z}^+$, then

$$\mathbf{z}^+\mathbf{M}_k\mathbf{z} = \omega_k\mathbf{z}^+\mathbf{z}$$

- We scan through U and use a root finder to locate the saddle point $d\omega/dk = 0$. We can then set the growth rate for that part of the pulse as $\Im(\omega)$.
- The Eady problem is marginal with respect to growth at the origin ($U = 0$) unless the flow at that boundary is westward. For most problems, you tend to find “convective” instability (growing downstream, decaying at a fixed point) rather than “absolute” instability (growing at a fixed point).

Non-zonal flows

The Arnold thm. generalizes for QG flows: the flow will be stable if

$$\frac{d\Psi}{d\mathcal{Q}} \geq 0 \quad \& \quad \left.\frac{d\Psi}{d\mathcal{B}}\right|_0 \geq 0 \quad \& \quad \left.\frac{d\Psi}{d\mathcal{B}}\right|_H \leq 0$$

everywhere. Again for zonal flows we can add translating coordinates or for circular flows, a rotating system. For those interested in the Hamiltonian formulation and the nonlinear versions, I'll include an appendix; it shows that this gives stability to finite amplitude disturbances.

Equilibration or turbulence

We are going to focus on the zonally symmetric problem. First consider the general setup (though we'll revert to the Eady problem for simplicity).

• Form the mean and fluctuating PV equations

$$\frac{\partial}{\partial t}\overline{q} + \frac{\partial}{\partial y}\overline{v'q'} = \overline{F}$$

$$\frac{\partial}{\partial t}q' + \overline{u}\frac{\partial}{\partial x}q' + v'\overline{q}_y + \nabla\cdot(\mathbf{u}'q') - \frac{\partial}{\partial y}\overline{v'q'} = F'$$

$$\left(\frac{\partial^2}{\partial y^2} + \mathcal{L}\right)\overline{u} = \beta - \overline{q}_y$$

$$\left(\nabla^2 + \mathcal{L}\right)\psi' = q'$$

and we've introduced forcing and damping terms. We pose it in a channel $y = 0$ to W.

- For weak waves, we can drop the terms quadratic in q' in the fluctuation equation; this is known as the "mean field approximation." Then it becomes the linear equation

  $$\frac{\partial}{\partial t}q' + \overline{u}\frac{\partial}{\partial x}q' + v'\overline{q}_y = F'$$

  But now the mean flow and mean PV gradient are altered by the diveregence of the eddy PV flux.
- For growing waves, the eddy flux is downgradient, but it must vanish at the north and south walls. If the PV gradient is positive, the flux will converge at the southern wall (increasing the PV near there) and diverge at the north (decreasing it). The net effect is to reduce the PV gradient. This applies for $-\theta$ at the upper boundary in the Eady problem; at the lower boundary, the PV gradient and fluxes are reversed. In both cases, the PV or thermal gradient is reduced and therefore the vertical shear is also. The eddies are driving the mean towards a more stable state.
- In turn, the shift in the mean state reduces the growth rate; in the presence of dissipation, the eddies can reach a neutral state in which the energy extracted from the mean balances the dissipation, while the eddy fluxes, together with the mean forcing and dissipation, set the final mean state.

• Eady example:

- Define the forcing of the mean top and bottom temperatures

  $$\frac{\partial}{\partial t}\overline{b} + \frac{\partial}{\partial y}\overline{v'b'} = -r(\overline{b} + y)$$

  $$\frac{\partial}{\partial t}b' + \overline{u}\frac{\partial}{\partial x}b' + \overline{b}_y\frac{\partial}{\partial x}\psi' = -rb'$$

- From the thermal variance equation

  $$\frac{\partial}{\partial t}\frac{1}{2}\overline{b'^2} + \overline{b}_y\overline{v'b'} = -r\overline{b'^2}$$

we can see that the eddies must flux heat downgradient to get growth or, in the final state, to balance losses. Consider the sketch of Eady instability; it shows the heat flux is indeed downgradient.

- For the standard problem with $b'(0) = \theta_0 \sin(\ell y) \exp(\imath kx) + c.c.$

$$\begin{pmatrix} v_0' \\ v_H' \end{pmatrix} = \imath k \begin{pmatrix} F_0(0) & F_H(0) \\ F_0(H) & F_H(H) \end{pmatrix} \begin{pmatrix} \theta_0 \\ \theta_H \end{pmatrix}$$

- To maximally simplify the analysis, let $K = 2.0650$ for which $-F_0(0) = F_H(H) = \frac{1}{2}$ and $-F_0(H) = F_H(0) = C = 0.12483$. Let $-U(0) = U(H) = \frac{1}{2}$. Then the growth rate is just $\sigma = kC$. Let $\ell = K/2$ so that $k = \sqrt{3}K/2$. For this scale, show that the growing eigenmode has $\theta_H = -\imath\theta_0$ — a 90° phase shift.
- Show that for $\theta_0 = A$

$$\overline{v'b'} = 2\sigma |A|^2 \sin^2 \ell y$$

on the boundaries. From this

$$\frac{\partial}{\partial t}\overline{b} + r\overline{b} = -ry - 2\sigma\ell |A|^2 \sin 2\ell y$$

This equilbrates at

$$\overline{b} = -y - \frac{2\sigma\ell}{r}|A|^2 \sin 2\ell y \quad \Rightarrow \quad \overline{b}_y = -1 - \frac{4\sigma\ell^2}{r}|A|^2 \cos 2\ell y$$

The temperature gradient is indeed reduced in the center of the channel and increased near the edge where, because of boundary condtions $v' = 0$, it is not effective at driving eddies.

- From the inversion equation for $\overline{\psi}$, we find the correction to the zonal flow

$$\begin{aligned} \overline{u}_0 &= -\frac{1}{2} + [F_0(0|2\ell) + F_H(0|2\ell)]\frac{4\sigma\ell^2}{r}|A|^2 \cos 2\ell y \\ &= -\frac{1}{2} - [\frac{1}{2} - C]\frac{4\sigma\ell^2}{r}|A|^2 \cos 2\ell y \end{aligned}$$

- We still do not know the amplitude, however. We can estimate it as follows:
  1) Since we've ignored wave-wave interactions, we have

$$b'(0) = \theta_0(y,t) \exp(\imath kx) + c.c.$$

  2) The variance equation gives

$$\frac{\partial}{\partial t}\frac{1}{2}|b'(0)|^2 + r|b'(0)|^2 = -\overline{v'b'}|_0\, \overline{b}_y(0)$$

or

$$\frac{\partial}{\partial t}|\theta_0|^2 + 2r|\theta_0|^2 = -(v_0\theta_0^* + v_0^*\theta_0)\, \overline{b}_y(0)$$

3) To get a rough amplitude equation, use the lowest order form $\theta_0 = A\sin(\ell y)$ and $v_0 = -\imath k(\frac{1}{2} + \imath C)\theta_0 \sin(\ell y)$ and integrate over y.

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 + 2r|A|^2 &= 2\sigma|A|^2 + \frac{8\sigma^2\ell^2}{r}|A|^4 \frac{\int \sin^2 \ell y \cos 2\ell y}{\int \sin^2 \ell y} \\ &= 2\sigma|A|^2 - \frac{4\sigma^2\ell^2}{r}|A|^4\end{aligned}$$

The amplitude equilibrates to $|A|^2 = (\sigma - r)r/2\sigma^2\ell^2$.

Generalized Eady model

We consider the case with $\mathcal{Q}_y = 0$ or

$$\mathcal{L}U(z) = \beta \quad , \quad \mathcal{L} = \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}$$

Then the interior equation

$$\frac{\partial}{\partial t}Q' + U(z)\frac{\partial}{\partial x}Q' + \psi'_x\mathcal{Q}_y + J(\psi', Q') = 0$$

has solutions with $Q' = 0$. For $\exp(\imath k(x - ct) + \imath\ell)$, we have

$$\mathcal{L}\psi' = |\mathbf{k}|^2\psi' = K^2\psi$$

The boundary conditions

$$\Theta_y = -U_z \quad , \quad \psi'_z = \theta'$$

(factors of f being incorporated into the theta's) when put in the conservation of θ on the boundary give

$$\frac{\partial}{\partial t}\theta' + U\frac{\partial}{\partial x}\theta' + \psi'_x\Theta_y + J(\psi', \theta') = 0 \quad \Rightarrow \quad (U - c)\psi_z - U_z\psi = 0$$

(dropping primes).

We can write the streamfunction in terms of the part forced by the lower boundary and that forced by the upper boundary

$$\psi' = \theta_0 F_0(z) + \theta_H F_H(z)$$

with

$$\mathcal{L}F = K^2 F \quad , \quad F_{0z}(0) = 1 \ , \ F_{0z}(H) = 0 \ , \ F_{Hz}(0) = 0 \ , \ F_{Hz}(H) = 1$$

Then we can write the dynamical equation in matrix form

$$\begin{pmatrix} U(0) - U_z(0)F_0(0) & -U_z(0)F_H(0) \\ -U_z(H)F_0(H) & U(H) - U_z(H)F_H(H) \end{pmatrix}\begin{pmatrix}\theta_0 \\ \theta_H\end{pmatrix} = c\begin{pmatrix}\theta_0 \\ \theta_H\end{pmatrix}$$

- Necessary condition: $M_{12}M_{21} < 0$. Since $F_0 < 0$ and $F_H > 0$ we must have

$$U_z(0)U_z(H) > 0$$

We can solve the $\mathcal{L}U$ equation

$$U_z = \frac{\beta}{f^2}\frac{N^2}{\overline{\rho}}\int_0^z \overline{\rho} + \frac{\overline{\rho}(0)N^2}{\overline{\rho}N^2(0)}U_z(0)$$

so that the shear at the top is

$$U_z(H) = \frac{\beta}{f^2}\frac{N^2(H)}{\overline{\rho}(H)}\int_0^H \overline{\rho} + \frac{\overline{\rho}(0)N^2(H)}{\overline{\rho}(H)N^2(0)}U_z(0)$$

The flow will be stable if

$$\frac{\overline{\rho}(H)}{N^2(H)}U_z(0)U_z(H) = U_z(0)\frac{\beta}{f^2}\int_0^H \overline{\rho} + \frac{\overline{\rho}(0)}{N^2(0)}U_z(0)^2 < 0$$

or

$$-\frac{\beta}{f^2}\frac{N^2(0)}{\overline{\rho}(0)}\int_0^H \overline{\rho} < U_z(0) < 0$$

Let us put that in terms of the net shear

$$U(H) - U(0) = \frac{\beta}{f^2}\int_0^H \frac{N^2}{\overline{\rho}}\int_0^z \overline{\rho} + \frac{\overline{\rho}(0)U_z(0)}{N^2(0)}\int_0^H \frac{N^2}{\overline{\rho}}$$

Multiplying the inequality by the coefficient of the $U_z(0)$ gives

$$-\frac{\beta}{f^2}\left(\int_0^H \overline{\rho}\right)\int_0^H \frac{N^2}{\overline{\rho}} < U(H) - U(0) - \frac{\beta}{f^2}\int_0^H \frac{N^2}{\overline{\rho}}\int_0^z \overline{\rho} < 0$$

or

$$-\frac{\beta}{f^2}\int_0^H \frac{N^2}{\overline{\rho}}\left(\int_0^H \overline{\rho} - \int_0^z \overline{\rho}\right) < U(H) - U(0) < \frac{\beta}{f^2}\int_0^H \frac{N^2}{\overline{\rho}}\int_0^z \overline{\rho} < 0$$

If we take the isothermal case with H the scale height, we have stability when

$$-\frac{1}{e}\beta\frac{N^2H^2}{f^2} < U(H) - U(0) < (e-1)\beta\frac{N^2H^2}{f^2}$$

Since $e - 1 = 1.7183$ and $1/e = 0.36788$ eastward shear can be much larger than westward shear. For an 8 km scale height, $NH/f = 1500\ km$. For $\beta = 2e - 11\ m^{-1}s^{-1}$, we find $\beta N^2H^2/f^2 = 45\ m/s$, and the bounds are $-16\ m/s < U(H) - U(0) < 77\ m/s$. For the ocean, we note that the integral on the left weights the deep water N^2 most heavily while the one on the right weights the upper water column value, which is usually larger. So we also expect westward shear will be less stable than eastward shear.

• Sufficient conditions? Is the necessary condition all you need assuming that K can be arbitrarily small or large? If we look at the equations for F, we see that large K will give a boundary-layer character, with the solutions decaying rapidly. Ie.,

$$F_0 \sim -\frac{f}{N(0)K}\exp(-KN(0)z/f)$$

implying the off diagonal terms are nearly zero and the eigenvalues are just the diagonal terms and are real. Indeed, $c \sim U(0)$ *or* $U(H)$ – the short waves are dominated by advection. In terms of the discriminant

$$\mathcal{D} = (M_{22} - M_{11})^2 + 4M_{12}M_{21}$$

the last term is very small and the first term goes to $[U(H) - U(0)]^2$ so that the $\mathcal{D}$ will be positive.

The case of small K is trickier. We actually need two terms in the approximation to F

$$F_0 = -A_0 \frac{1}{K^2} + \tilde{F}_0(z)$$

with

$$\mathcal{L}\tilde{F}_0 = -A_0$$

Integrating this and applying the shear boundary condition gives

$$A_0 = \frac{\overline{\rho}(0) f^2}{N^2(0) \int_0^H \overline{\rho}}$$

Using just the order K^{-2} terms in the matrix implies the determinant is 0 and the trace is

$$Tr = \frac{U_z(0)A_0 - U_z(H)A_H}{K^4}$$

For the $\beta = 0$ case, the expression for the shear shows that the trace is also zero, so the discriminant $\mathcal{D} = Tr^2 - 4Det = 0$ and we need to go to higher order. This calculation will show that $\mathcal{D}$ is negative all the way to zero wavenumber.

But, in fact, all we really need to do is prove that $M_{22} - M_{11}$ passes through zero at some K since $\mathcal{D}(K)$ will be negative there. Let's take the case $U_z(0), U_z(H), U(H) > 0$ and $U(0) = 0$. Then

$$M_{22} - M_{11} = U(H) - U_z(H)F_H(H) - U(0) + U_z(0)F_0(0)$$

is clearly positive for large K when the F's vanish, but for small K becomes

$$M_{22} - M_{11} \sim U(H) - U_z(H)A_H/K^2 - U(0) - U_z(0)A_0/K^2$$

which will become negative. Thus there will be a critical wavenumber at which

$$\mathcal{D} = 4M_{12}M_{21} < 0$$

and nearby wavenumbers will also be unstable.

- For the CONST case,

$$F_0 = -\frac{\cosh(K'[H-z])}{K' \sinh(K'H)} \quad , \quad F_H = \frac{\cosh(K'z)}{K' \sinh(K'H)} \quad , \quad K' = KN/f$$

$$F_H(H) = -F_0(0) = \frac{\cosh(K'H)}{K' \sinh(K'H)}$$

$$F_H(0) = -F_0(H) = \frac{1}{K' \sinh(K'H)}$$