Roadmap #4: Beyond QG

Frontogenesis

Fronts are regions of very rapid horizontal temperature change and, since the along-front winds are still pretty geostrophic, strong wind shear. Their Rossby numbers frequently become order one or larger, so we want to explore them as an example of non-QG dynamics. We shall look at the problem of how fronts are generated using QG and then semigeostrophic models.

- But first, examine what can cause intensification of the temperature gradient.
  - From the thermodynamic equation, calculate the rate of increase of $\nabla|\theta|^2$:

$$(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j})\frac{1}{2}\left(\frac{\partial \theta}{\partial x_i}\right)^2 + \frac{\partial \theta}{\partial x_i}\frac{\partial u_j}{\partial x_i}\frac{\partial \theta}{\partial x_j} = 0$$

or

$$\frac{1}{|\nabla\theta|}(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j})|\nabla\theta| = -\frac{1}{|\nabla\theta|^2}\frac{\partial \theta}{\partial x_i}\frac{\partial u_j}{\partial x_i}\frac{\partial \theta}{\partial x_j}$$

  - Split the tensor $\frac{\partial u_j}{\partial x_i}$ into the trace times the identity (but that's zero by continuity), an antisymmetric part (related to the vorticity), and the rate-of-strain tensor, the symmetric part.
  - show that only the symmtric part affects the r.h.s. Therefore the gradient grows at a rate given by the most negative eigenvalue of

$$\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

  Since the trace is zero, the sum of the eigenvalues is zero; since the matrix is symmetric, they are real, so you expect to have some positive and some negative values. The negative values correspond to the direction where the θ contours are being pushed close together, with the fluid between them being squirted out parallel to the contours.

For the QG system, this is the basic phenomenon: the gradient in θ grows exponentially and the thermal wind implies the shear does also. For the full equations or the semi-geostrophic eqns., the magnitude of the eigenvalue also increases, so that the growth is super-exponential and the front becomes singular in a finite time.

- Now consider an example with $u \to -Dx + u(x,z,t)$, $v \to Dy + v(x,z,t)$, $p \to fDxy - \frac{1}{2}D^2(x^2+y^2) + \frac{1}{2}N^2z^2 + p(x,z,t)$ and $b \to N^2z + b$ in the Boussinesq model (constant N^2

will be used).

$$\frac{D}{Dt}u - Du - fv = -p_x$$
$$\frac{D}{Dt}v + Dv + fu = 0$$
$$\frac{D}{Dt}w = -p_z + b$$
$$\frac{D}{Dt}b + wN^2 = 0$$
$$u_x + w_z = 0$$
$$\frac{D}{Dt} = \frac{\partial}{\partial t} - Dx\frac{\partial}{\partial x} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}$$

- show that the angular momentum $M = v + fx$ satisfies

$$\frac{D}{Dt}M = -DM \quad \Rightarrow \quad \frac{D}{Dt}M\exp(Dt) = 0$$

- we already have conservation of temperature

$$\frac{D}{Dt}(b + N^2 z) = 0 \quad or \quad \frac{D}{Dt}b_T = 0$$

- we also conserve Ertel PV

$$q = (v_x + f)(N^2 + b_z) - v_z b_x$$

prove this is still conserved with the equations above.

- note that

$$q = \frac{\partial(M, b_T)}{\partial(x, z)}$$

– $1/q$ represents the area between M and b_T contours. Conservation of q implies that the area of a patch formed by two M and two b_T contours is fixed. The latter are material lines, but the former are not.

- QG form
  - Find the vertical vorticity eqn by $\frac{\partial}{\partial x}$ of the v eqn

$$\frac{D}{Dt}\zeta + w_x v_z = (f + \zeta)w_z \quad , \quad \zeta = v_x$$

In QG, $u_g = 0$, $\zeta << f$, and $w\frac{\partial}{\partial z}$ is ignored

$$(\frac{\partial}{\partial t} - Dx\frac{\partial}{\partial x})\zeta = fw_z \quad , \quad \zeta = v_x$$

  - The QG form of the u equation just gives geostrophy

$$fv = p_x = f\psi_x$$

- The hydrostatic and buoyancy equations give

$$b = p_z = f\psi_z \quad , \quad (\frac{\partial}{\partial t} - Dx\frac{\partial}{\partial x})b + N^2 w = 0$$

in the QG limit. Combining these shows that

$$(\frac{\partial}{\partial t} - Dx\frac{\partial}{\partial x})Q = 0 \quad , \quad Q = \frac{\partial^2}{\partial x^2}\psi + \mathcal{L}\psi \quad , \quad \mathcal{L} = \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}$$

- In the QG Eady-like problem, $Q = 0$ and the boundary temperature just behaves like

$$(\frac{\partial}{\partial t} - Dx\frac{\partial}{\partial x})b = 0$$

If we start with $b = b_0 \tanh(x/L)$, the solution is

$$b(x, 0, t) = b_0 \tanh(x \exp(Dt)/L)$$

Thus a front (but not discontinuity) forms due to the large scale strain field.

- To examine the flow everywhere, assuming an unbounded system with constant N^2, we let $x = X\exp(-Dt)$, $z = Z(f/N)\exp(-Dt)$, $\psi = \Psi\exp(-Dt)$. The PV equation and the boundary condition become

$$\Psi_{XX} + \Psi_{ZZ} = 0 \quad \text{with} \quad \Psi_Z = b_0(X)/N$$

Therefore Ψ and $v = \Psi_X$ and $b' = N\Psi_Z$ are time-independent in the X, Z coordinates and simply contract vertically and horizontally with time. On the other hand, the vorticity $v_x = \exp(Dt)\Psi_{XX}$ grows exponentially with time. Note that if b_0 is antisymmetric around the origin, v will be symmetric and ζ antisymmetric: the front is not tilted in the vertical. As an explicit, example take a sinusoidal boundary condition; the solution (putting the squeezing into the wavenumbers rather than the coordinates) is

$$\begin{aligned}
&b(x, 0, t) = b_0 \sin(Kx) \quad , \quad K = k\exp(Dt) \\
&\quad \Rightarrow \\
&b = b_0 \sin(kX)\exp(-K_z Z) \quad , \quad K_z = k(N/f)\exp(Dt) \\
&\psi = -\frac{b_0}{fK_z}\sin(Kx)\exp(-K_z z) \\
&v = -\frac{b_0 K}{fK_z}\cos(Kx)\exp(-K_z z) \\
&\zeta = \frac{b_0 K^2}{fK_z}\sin(Kx)\exp(-K_z z)
\end{aligned}$$

- Most studies have a lid or variable $N^2(z)$; in that case, the transformation to Z cannot be done and the v velocity will no longer be constant: in the case above, the shear is very small initially, but it extends to great depths. On the other hand, the remarks about symmetry remain true.

- For an analytic example with a lid, use

$$b(x,0,t) = b_0 \sin(kx \exp(Dt)) = b_0 \sin(Kx) \quad , \quad b(x,H,t) = 0$$

But this is worse:

$$v = -\frac{b_0 K}{f K_z} \cos(Kx) \frac{\cosh(K_z[H-z])}{\sinh(K_z H)}$$

gives $v(0)$ which blows up as $t << 0$. For small K, the vertical curvature is small, so to match the dissimilar boundary conditions a large velocity is required. If, however, we assume the initial b is the same on the top and bottom, then

$$b = b_0 \sin(Kx) \frac{\cosh(K_z[z-H/2])}{\cosh(K_z H/2)}$$
$$\psi = \frac{b_0}{f K_z} \sin(Kx) \frac{\sinh(K_z[z-H/2])}{\cosh(K_z H/2)}$$
$$v = \frac{b_0 K}{f K_z} \cos(Kx) \frac{\sinh(K_z[z-H/2])}{\cosh(K_z H/2)}$$
$$\zeta = -\frac{b_0 K^2}{f K_z} \sin(Kx) \frac{\sinh(K_z[z-H/2])}{\cosh(K_z H/2)}$$

Now, as $t \to -\infty$

$$v(x,0) = -\frac{b_0 K}{f K_z} \cos(Kx) \tanh(K_z H/2) \to -\frac{b_0 K H}{2f} \cos(Kx)$$

which vanishes. As $t \to +\infty$,

$$v(x,0) \to -\frac{b_0 K}{f K_z} \cos(Kx)$$

For this case, the velocity reaches a limit, but the vorticity continues to grow exponentially as the scale shrinks.

$$\zeta(x,0) \to \frac{b_0 K^2}{f K_z} \sin(Kx)$$

- Non-QG: the full system includes a y-vorticity equation for $\nabla^2\phi$ with $u = \phi_z, \; w = -\phi_x$

$$\left(\frac{D}{Dt} - D\right) \nabla^2 \phi = f v_z - b_x$$
$$\left(\frac{D}{Dt} + D\right) v = -f \phi_z$$
$$\frac{D}{Dt} b = \phi_x N^2$$
$$\frac{D}{Dt} = \frac{\partial}{\partial t} - Dx \frac{\partial}{\partial x} + \phi_z \frac{\partial}{\partial x} - \phi_x \frac{\partial}{\partial z}$$

- Show that a positive imbalance $fv_z > b_x$ induces a flow which tends to reduce v_z by Coriolis torques and increase b_x by vertical movement of the basic stratification.

• Semigeostrophic approx: This assumes the along-front scale is large and the cross-front velocity u is small compared to v. Then v remains geostrophic and b is hydrostatic (still with the $N^2 z$ term factored out). We are actually assuming that the horizontal adjustment process is extremely effective.

- Now

$$\frac{q}{N^2} - f = \psi_{xx} + \frac{f^2}{N^2}\psi_{zz} + \frac{f}{N^2}(\psi_{xx}\psi_{zz} - \psi_{xz}^2)$$

The QG form has just the linear terms. This can also be written as

$$\frac{q}{f} = \frac{\partial(\Psi_x, \Psi_z)}{\partial(x,z)} \quad , \quad \Psi = \psi + f\frac{x^2}{2} + \frac{N^2}{f}\frac{z^2}{2}$$

which is the 2D Monge-Ampère equation (that actually does help because a fair amount is known about its properties).

- Consider a constant PV case $q = fN^2$. Then

$$\psi_{xx} + \frac{f^2}{N^2}\psi_{zz} + \frac{f}{N^2}(\psi_{xx}\psi_{zz} - \psi_{xz}^2) = 0$$

at all times and we only need to worry about the boundary

$$\frac{D}{Dt}\psi_z = 0$$

- But... The inversion is nonlinear and the advection of ψ_z depends on u also. We can find the ageostrophic circulation from the omega equation

$$f\frac{\partial}{\partial z}[\frac{D}{Dt}v + Dv] - \frac{\partial}{\partial x}[\frac{D}{Dt}b] = -f^2\phi_{zz} - N^2\phi_{xx}$$

$$fJ(\phi_z, v) - Dfv_z - Db_x - J(\phi_x, b) = f^2\phi_{zz} + N^2\phi_{xx}$$

$$f^2\phi_{zz} + N^2\phi_{xx} + fJ(\phi_x, \psi_z) + fJ(\psi_x, \phi_z) = -2Db_x$$

or

$$f(f + \psi_{xx})\phi_{zz} + (N^2 + f\psi_{zz})\phi_{xx} - 2f\psi_{xz}\phi_{xz} = -2Df\psi_{xz}$$

• Geostrophic coordinates: For constant PV

$$\frac{\partial(M/f, b_T)}{\partial(x,z)} = 1$$

so that the area between two M/f and two b_T surfaces is preserved. We could try to use these as new coordinates; however the first is not conserved.

- So let's just change the X coordinate to $M/f = x + v/f$ and use the same $Z = z$ coordinate but with $\frac{\partial}{\partial Z}$ indicating the derivative at constant X (Hoskins and Bretherton, 1972). Then

$$\frac{\partial}{\partial x} = \frac{\zeta_a}{f} \frac{\partial}{\partial X}$$

where ζ_a is the absolute vorticity $f + \zeta$. Applying this to v gives

$$\zeta_a - f = \frac{\zeta_a}{f} v_X \quad \Rightarrow \quad \zeta_a = \frac{f}{1 - v_X/f}$$

The absolute vorticity can blow up in a finite time if $v_X \to f$.

- Consider the thermal wind

$$f M_z = b_x \quad \Rightarrow \quad f \frac{\partial(x, M)}{\partial(x, z)} = \frac{\partial(b, z)}{\partial(x, z)} \quad \Rightarrow \quad f \frac{\partial(x, M)}{\partial(X, Z)} = \frac{\partial(b, Z)}{\partial(X, Z)}$$

$$f \, det \begin{pmatrix} 1 - v_X/f & f \\ -v_Z/f & 0 \end{pmatrix} = f v_Z = b'_X$$

so we have thermal wind and

$$v = \Psi_X \quad , \quad b' = f \Psi_Z \quad , \quad \Psi = \psi + v^2/f$$

- Note on Jacobians:

$$\frac{\partial(A, B)}{\partial(X, Z)} = \frac{\partial(A, B)}{\partial(x, z)} \frac{\partial(x, y)}{\partial(X, Z)} \quad \text{since} \quad \frac{\partial(A, B)}{\partial(X, Z)} = det \begin{pmatrix} A_X & B_X \\ A_Z & B_Z \end{pmatrix}$$

- For the $q = fN^2$ case, the PV equation

$$\frac{\partial(M/f, b_T/N^2)}{\partial(X, Z)} \Big/ \frac{\partial(x, z)}{\partial(X, Z)} = 1 \quad or \quad \frac{\partial(X, b_T/N^2)}{\partial(X, Z)} = \frac{\partial(x, z)}{\partial(X, Z)}$$

since $M/f = X$. The lhs is

$$1 + b_Z/N^2 = 1 + \Psi_{zz}/N^2$$

and the rhs is

$$det \begin{pmatrix} 1 - v_X/f & -v_Z/f \\ 0 & 1 \end{pmatrix} = 1 - v_X/f = 1 - \Psi_{XX}/f$$

Together, these give

$$\Psi_{XX} + \frac{f^2}{N^2} \Psi_{ZZ} = 0$$

In geostrophic coordinates, the PV equation is linear and isomorphic to the QG PV eqn.

- To find b at the ground, use conservation of $b(x,0,t)$ and of Xe^{Dt} (from the M equation); together these imply

$$b(X,0,t) = b(Xe^{Dt},0,0) = B_0(Xe^{Dt})$$

again identical to the QG problem.

- So we solve the QG equations and find the fields as function of X, Z for a given t by taking $f\Psi_Z(X,0,t) = B_0(Xe^{Dt})$ and inverting the linear PV equation just as in the Eady edge wave problem (but without waves). Then we translate back using

$$x = X - \Psi_X(X,Z) \ , \quad z = Z$$

and look at the solution in physical space.

- If we take the sinusoidal model in the semi-infinite domain

$$v = -\frac{b_0}{N}\cos(KX)\exp(-K_z Z)$$

and we can draw curves of $v(x,0,t)$ easily.

- a Balance models of the general circulation of the ocean
- b Interpretation of atmospheric observations/analyses
- c Jets and eddies
- d Data assimilation
- e Loss of balance
- f Ripa's thm