Roadmap #4: Beyond QG

Frontogenesis

Fronts are regions of very rapid horizontal temperature change and, since the along-front winds are still pretty geostrophic, strong wind shear. Their Rossby numbers frequently become order one or larger, so we want to explore them as an example of non-QG dynamics. We shall look at the problem of how fronts are generated using QG and then semigeostrophic models.

- But first, examine what can cause intensification of the temperature gradient.
  - From the thermodynamic equation, calculate the rate of increase of $\nabla|\theta|^2$:

  \[
  \left( \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right) \frac{1}{2} \left( \frac{\partial \theta}{\partial x_i} \right)^2 + \frac{\partial \theta}{\partial x_i} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = 0
  \]

  or

  \[
  \frac{1}{|\nabla \theta|} \left( \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right) |\nabla \theta| = -\frac{1}{|\nabla \theta|^2} \frac{\partial \theta}{\partial x_i} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}
  \]

  - Split the tensor $\frac{\partial u_j}{\partial x_i}$ into the trace times the identity (but that’s zero by continuity), an antisymmetric part (related to the vorticity), and the rate-of-strain tensor, the symmetric part.
  - show that only the symmetric part affects the r.h.s. Therefore the gradient grows at a rate given by the most negative eigenvalue of

  \[
  \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]
  \]

  Since the trace is zero, the sum of the eigenvalues is zero; since the matrix is symmetric, they are real, so you expect to have some positive and some negative values. The negative values correspond to the direction where the $\theta$ contours are being pushed close together, with the fluid between them being squirted out parallel to the contours. For the QG system, this is the basic phenomenon: the gradient in $\theta$ grows exponentially and the thermal wind implies the shear does also. For the full equations or the semi-geostrophic eqns., the magnitude of the eigenvalue also increases, so that the growth is super-exponential and the front becomes singular in a finite time.

- Now consider an example with $u \to -Dx + u(x,z,t)$, $v \to Dy + v(x,z,t)$, $p \to fDxy - \frac{1}{2}D^2(x^2 + y^2) + \frac{1}{2}N^2z^2 + p(x,z,t)$ and $b \to N^2z + b$ in the Boussinesq model (constant $N^2$...
\[ \frac{D}{Dt} u - Du - fv = -p_x \]
\[ \frac{D}{Dt} v + Dv + fu = 0 \]
\[ \frac{D}{Dt} w = -p_z + b \]
\[ \frac{D}{Dt} b + wN^2 = 0 \]
\[ u_x + w_z = 0 \]

- show that the angular momentum \( M = v + fx \) satisfies

\[ \frac{D}{Dt} M = -DM \Rightarrow \frac{D}{Dt} M \exp(Dt) = 0 \]

- we already have conservation of temperature

\[ \frac{D}{Dt} (b + N^2 z) = 0 \quad \text{or} \quad \frac{D}{Dt} b_T = 0 \]

- we also conserve Ertel PV

\[ q = (v_x + f) (N^2 + b_z) - v_z b_x \]

prove this is still conserved with the equations above.

- note that

\[ q = \frac{\partial(M, b_T)}{\partial(x, z)} \]

\(- 1/q\) represents the area between \( M \) and \( b_T \) contours. Conservation of \( q \) implies that the area of a patch formed by two \( M \) and two \( b_T \) contours is fixed. The latter are material lines, but the former are not.

- QG form

- Find the vertical vorticity eqn by \( \frac{\partial}{\partial z} \) of the \( v \) eqn

\[ \frac{D}{Dt} \zeta + w_x v_z = (f + \zeta) w_z \quad , \quad \zeta = v_x \]

In QG, \( u_g = 0 \), \( \zeta << f \), and \( w \frac{\partial}{\partial z} \) is ignored

\[ (\frac{\partial}{\partial t} - Dx \frac{\partial}{\partial x}) \zeta = f w_z \quad , \quad \zeta = v_x \]

- The QG form of the \( u \) equation just gives geostrophy

\[ fv = p_x = f \psi_x \]
The hydrostatic and buoyancy equations give

\[ b = p_z = f \psi_z \quad , \quad (\frac{\partial}{\partial t} - D_x \frac{\partial}{\partial x})b + N^2 w = 0 \]

in the QG limit. Combining these shows that

\[ (\frac{\partial}{\partial t} - D_x \frac{\partial}{\partial x})Q = 0 \quad , \quad Q = \frac{\partial^2}{\partial x^2} \psi + \mathcal{L} \psi \quad , \quad \mathcal{L} = \frac{f^2}{N^2} \frac{\partial^2}{\partial z^2} \]

In the QG Eady-like problem, \( Q = 0 \) and the boundary temperature just behaves like

\[ (\frac{\partial}{\partial t} - D_x \frac{\partial}{\partial x})b = 0 \]

If we start with \( b = b_0 \tanh(x/L) \), the solution is

\[ b(x,0,t) = b_0 \tanh(x \exp(Dt)/L) \]

Thus a front (but not discontinuity) forms due to the large scale strain field.

To examine the flow everywhere, assuming an unbounded system with constant \( N^2 \), we let \( x = X \exp(-Dt), z = Z(f/N) \exp(-Dt), \psi = \Psi \exp(-Dt) \). The PV equation and the boundary condition become

\[ \Psi_{XX} + \Psi_{ZZ} = 0 \quad \text{with} \quad \Psi_Z = b_0(X)/N \]

Therefore \( \Psi \) and \( v = \Psi_X \) and \( b' = N\Psi_Z \) are time-independent in the \( X, Z \) coordinates and simply contract vertically and horizontally with time. On the other hand, the vorticity \( v_x = \exp(Dt)\Psi_{XX} \) grows exponentially with time. Note that if \( b_0 \) is antisymmetric around the origin, \( v \) will be symmetric and \( \zeta \) antisymmetric: the front is not tilted in the vertical. As an explicit, example take a sinusoidal boundary condition; the solution (putting the squeezing into the wavenumbers rather than the coordinates) is

\[ b(x,0,t) = b_0 \sin(Kx) \quad , \quad K = k \exp(Dt) \]

\[ \Rightarrow \]

\[ b = b_0 \sin(kX) \exp(-K_z Z) \quad , \quad K_z = k(N/f) \exp(Dt) \]

\[ \psi = -\frac{b_0}{fK_z} \sin(Kx) \exp(-K_z z) \]

\[ v = -\frac{b_0K}{fK_z} \cos(Kx) \exp(-K_z z) \]

\[ \zeta = \frac{b_0K^2}{fK_z} \sin(Kx) \exp(-K_z z) \]

Most studies have a lid or variable \( N^2(z) \); in that case, the transformation to \( Z \) cannot be done and the \( v \) velocity will no longer be constant: in the case above, the shear is
very small initially, but it extends to great depths. On the other hand, the remarks about symmetry remain true.

- For an analytic example with a lid, use

\[ b(x, 0, t) = b_0 \sin(kx \exp(Dt)) = b_0 \sin(Kx) \quad , \quad b(x, H, t) = 0 \]

But this is worse:

\[ v = -\frac{b_0 K}{fK_z} \cos(Kx) \frac{\cosh(K_z[H - z])}{\sinh(K_zH)} \]

gives \( v(0) \) which blows up as \( t \ll 0 \). For small \( K \), the vertical curvature is small, so to match the dissimilar boundary conditions a large velocity is required. If, however, we assume the initial \( b \) is the same on the top and bottom, then

\[ b = b_0 \sin(Kx) \frac{\cosh(K_z[z - H/2])}{\cosh(K_zH/2)} \]

\[ \psi = \frac{b_0}{fK_z} \sin(Kx) \frac{\sinh(K_z[z - H/2])}{\cosh(K_zH/2)} \]

\[ v = \frac{b_0 K}{fK_z} \cos(Kx) \frac{\sinh(K_z[z - H/2])}{\cosh(K_zH/2)} \]

\[ \zeta = -\frac{b_0 K^2}{fK_z} \sin(Kx) \frac{\sinh(K_z[z - H/2])}{\cosh(K_zH/2)} \]

Now, as \( t \to -\infty \)

\[ v(x, 0) = -\frac{b_0 K}{fK_z} \cos(Kx) \tanh(K_zH/2) \to -\frac{b_0 KH}{2f} \cos(Kx) \]

which vanishes. As \( t \to +\infty \),

\[ v(x, 0) \to -\frac{b_0 K}{fK_z} \cos(Kx) \]

For this case, the velocity reaches a limit, but the vorticity continues to grow exponentially as the scale shrinks.

\[ \zeta(x, 0) \to \frac{b_0 K^2}{fK_z} \sin(Kx) \]

- Non-QG: the full system includes a \( y \)-vorticity equation for \( \nabla^2 \phi \) with \( u = \phi_z, \ w = -\phi_x \)

\[
\left( \frac{D}{Dt} - D \right) \nabla^2 \phi = f v_z - b_x \\
\left( \frac{D}{Dt} + D \right) v = -f \phi_z \\
\frac{D}{Dt} b = \phi_x N^2 \\
\frac{D}{Dt} \frac{\partial}{\partial t} - D x \frac{\partial}{\partial x} + \phi_z \frac{\partial}{\partial x} - \phi_x \frac{\partial}{\partial z}
\]
- Show that a positive imbalance $fv_z > bx$ induces a flow which tends to reduce $v_z$ by Coriolis torques and increase $bx$ by vertical movement of the basic stratification.

- Semigeostrophic approx: This assumes the along-front scale is large and the cross-front velocity $u$ is small compared to $v$. Then $v$ remains geostrophic and $b$ is hydrostatic (still with the $N^2z$ term factored out). We are actually assuming that the horizontal adjustment process is extremely effective.

- Now
  $$\frac{q}{N^2} - f = \psi_{xx} + \frac{f^2}{N^2} \psi_{zz} + \frac{f}{N^2} ( \psi_{xx} \psi_{zz} - \psi^2_{zz} )$$

  The QG form has just the linear terms. This can also be written as
  $$\frac{q}{\bar{f}} = \frac{\partial (\Psi, \Psi_z)}{\partial (x, z)}$$

  which is the 2D Monge-Ampère equation (that actually does help because a fair amount is known about its properties).

- Consider a constant PV case $q = fN^2$. Then
  $$\psi_{xx} + \frac{f^2}{N^2} \psi_{zz} + \frac{f}{N^2} ( \psi_{xx} \psi_{zz} - \psi^2_{zz} ) = 0$$

  at all times and we only need to worry about the boundary
  $$\frac{D}{Dt} \psi_z = 0$$

- But... The inversion is nonlinear and the advection of $\psi_z$ depends on $u$ also. We can find the ageostrophic circulation from the omega equation
  $$f \frac{\partial}{\partial z} \left[ \frac{D}{Dt} v + Dv \right] - \frac{\partial}{\partial x} \left[ \frac{D}{Dt} b \right] = -f^2 \phi_{zz} - N^2 \phi_{xx}$$

  $$fJ(\phi_z, v) - Dfv_z - Db_x - J(\phi_x, b) = f^2 \phi_{zz} + N^2 \phi_{xx}$$

  $$f^2 \phi_{zz} + N^2 \phi_{xx} + fJ(\phi_x, \psi_z) + fJ(\psi_x, \phi_z) = -2Db_x$$

  or
  $$f(f + \psi_{xx}) \phi_{zz} + (N^2 + f\psi_{zz}) \phi_{xx} - 2f \psi_{xz} \phi_{xz} = -2Df \psi_{xz}$$

- Geostrophic coordinates: For constant PV
  $$\frac{\partial (M/f, b_T)}{\partial (x, z)} = 1$$

  so that the area between two $M/f$ and two $b_T$ surfaces is preserved. We could try to use these as new coordinates; however the first is not conserved.
- So let’s just change the $X$ coordinate to $M/f = x + v/f$ and use the same $Z = z$ coordinate but with $\frac{\partial}{\partial Z}$ indicating the derivative at constant $X$ (Hoskins and Bretherton, 1972). Then
\[
\frac{\partial}{\partial x} = \frac{\zeta_a}{f} \frac{\partial}{\partial X}
\]
where $\zeta_a$ is the absolute vorticity $f + \zeta$. Applying this to $v$ gives
\[
\zeta_a - f = \frac{\zeta_a}{f} v_X \Rightarrow \zeta_a = \frac{f}{1 - v_X/f}
\]
The absolute vorticity can blow up in a finite time if $v_X \to f$.

- Consider the thermal wind
\[
f M_z = b_x \Rightarrow f \frac{\partial(x, M)}{\partial(x, z)} = \frac{\partial(b, z)}{\partial(x, z)} \Rightarrow f \frac{\partial(x, M)}{\partial(X, Z)} = \frac{\partial(b, Z)}{\partial(X, Z)}
\]
\[
f \det \begin{pmatrix} 1 - v_X/f & f \\ -v_Z/f & 0 \end{pmatrix} = f v_Z = b'_X
\]
so we have thermal wind and
\[
v = \Psi_X , \quad b' = f \Psi_Z , \quad \Psi = \psi + v^2/f
\]

- Note on Jacobians:
\[
\frac{\partial(A, B)}{\partial(X, Z)} = \frac{\partial(A, B)}{\partial(x, z)} \frac{\partial(x, y)}{\partial(X, Z)} \quad \text{since} \quad \frac{\partial(A, B)}{\partial(X, Z)} = \det \begin{pmatrix} A_X & B_X \\ A_Z & B_Z \end{pmatrix}
\]

- For the $q = fN^2$ case, the PV equation
\[
\frac{\partial(M/f, b_T/N^2)}{\partial(X, Z)} \bigg/ \partial(x, z) \frac{\partial(x, z)}{\partial(X, Z)} = 1 \quad \text{or} \quad \frac{\partial(X, b_T/N^2)}{\partial(X, Z)} = \frac{\partial(x, z)}{\partial(X, Z)}
\]
since $M/f = X$. The lhs is
\[
1 + b_Z/N^2 = 1 + \Psi_{zz}/N^2
\]
and the rhs is
\[
det \begin{pmatrix} 1 - v_X/f & -v_Z/f \\ 0 & 1 \end{pmatrix} = 1 - v_X/f = 1 - \Psi_{XX}/f
\]
Together, these give
\[
\Psi_{XX} + \frac{f^2}{N^2} \Psi_{ZZ} = 0
\]
In geostrophic coordinates, the PV equation is linear and isomorphic to the QG PV eqn.
- To find $b$ at the ground, use conservation of $b(x,0,t)$ and of $Xe^{Dt}$ (from the $M$ equation); together these imply

$$b(X,0,t) = b(Xe^{Dt},0,0) = B_0(Xe^{Dt})$$

again identical to the QG problem.

- So we solve the QG equations and find the fields as function of $X,Z$ for a given $t$ by taking $f\Psi_Z(X,0,t) = B_0(Xe^{Dt})$ and inverting the linear PV equation just as in the Eady edge wave problem (but without waves). Then we translate back using

$$x = X - \Psi_X(X,Z), \quad z = Z$$

and look at the solution in physical space.

- If we take the sinusoidal model in the semi-infinite domain

$$v = -\frac{b_0}{N} \cos(KX) \exp(-KzZ)$$

and we can draw curves of $v(x,0,t)$ easily.

- a Balance models of the general circulation of the ocean
- b Interpretation of atmospheric observations/analyses
- c Jets and eddies
- d Data assimilation
- e Loss of balance
- f Ripa’s thm