Flow over topography

Consider the following QG form for depth $H - h'$

$$\frac{\partial}{\partial t} q + \mathbf{u} \cdot \nabla q = 0$$

$$\mathbf{u} = \hat{\mathbf{z}} \times \nabla \psi$$

$$q = \nabla^2 \psi + \beta y + \frac{f_0}{H} h'$$

For steady flow

$$q = Q(\psi)$$

and for an upstream flow $\psi \to -Uy$ and $h' = 0$, we have

$$Q(\psi) = -\frac{\beta}{U} \psi$$

which holds on all streamlines communicating with ∞. Substituting

$$\nabla^2 \psi' + \frac{\beta}{U} \psi' = -\frac{f_0}{H} h'$$

with $\psi = -Uy + \psi'$. Note that solutions to this linear equation will still be solutions to the full problem.

We will deal with a simple case in a channel with width W

$$h' = h(x) \sin \ell y \quad , \quad \ell = \frac{\pi}{W}$$

and

$$\psi' = \psi'(x) \sin \ell y$$

Then

$$\mathcal{L} \psi' = -\frac{f_0}{H} h'$$

with the operator

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} - \ell^2 + \frac{\beta}{U}$$

Greens' functions

We can solve this kind of problem by defining

$$\mathcal{L}G(x|x') = \delta(x - x')$$

giving

$$\psi' = -\int dx' G(x|x') \frac{f_0}{H} h'(x')$$

For very narrow topography

$$\psi' \simeq -\frac{f_o}{H} A G(x|x')$$

where $A = \int dx' h'(x')$ being the area under the ridge. So G gives us a pretty good idea of the response.

Because the operator is second order, G must be continuous at $x = x'$ but its first derivative will not be. With this operator

$$G'(x' + \epsilon, x') - G'(x' - \epsilon, x') \to 1$$

The Greens' function can be written in terms of a solution $\phi^+(x)$ which satisfies the boundary condition as $x \to \infty$ and $\phi^-(x)$ for the left boundary condition. Then

$$G(x|x') = G_0 \begin{cases} \phi^-(x)\phi^+(x') & x < x' \\ \phi^-(x')\phi^+(x) & x > x' \end{cases}$$

and applying the matching condition on the derivative gives

$$G_0 = 1 \Big/ \left[\phi^- \frac{\partial}{\partial x} \phi^+ - \phi^+ \frac{\partial}{\partial x} \phi^- \right]$$

if $U < 0$ we have

$$\phi^\pm = \exp(\mp kx)$$

with

$$k = \sqrt{\ell^2 + \frac{\beta}{U}}$$

and

$$G = -\frac{1}{2k} \exp(-k|x - x'|)$$

But for $U > 0$ there can be sinusoidal waves with

$$k = \sqrt{\frac{\beta}{U} - \ell^2}$$

and

$$\phi^- = 0 \quad , \quad \phi^+ = \sin(kx)$$

giving

$$G(x|x') = \begin{cases} 0 & x < x' \\ \frac{1}{k} \sin k(x - x') & x > x' \end{cases}$$

and we have a downstream lee wave with

$$\psi = -\frac{f_0 A}{Hk} \sin kx$$

Lee wave

Why the asymmetry? Why not sines or cosines fro $x < x'$ as well? Here we cannot apply the boundary condition $\psi' \to 0$; we need a radiation condition. No energy should be coming in from $-\infty$ and it should be going out at $+\infty$.

Group velocity

$$\omega = Uk - \frac{\beta k}{k^2 + \ell^2}$$

gives

$$c_g = U + \beta \frac{k^2 - \ell^2}{(k^2 + \ell^2)^2}$$

(westward for long waves, eastward for short waves when $U = 0$). For the stationary wave, though

$$c_g = U + U\frac{k^2 - \ell^2}{k^2 + \ell^2} = U\frac{2k^2}{k^2 + \ell^2} > 0$$

So waves on the left side would not satisfy the radiation condition.

Initial value problem

For the linearized initial value problem

$$\frac{\partial}{\partial t}q' + \frac{\partial}{\partial x}\left[Uq' + \beta\psi' + U\frac{f_0 h'}{H}\right] = 0$$

with

$$\left[\frac{\partial^2}{\partial x^2} - \ell^2\right]\psi' = q'$$

If we write the Laplace transform solution

$$\hat{q}(x) = \int_0^\infty e^{-st} q(x,t)$$

then

$$s\hat{q} + \frac{\partial}{\partial x}\left[U\hat{q} + \beta\hat{\psi}\right] = -\frac{1}{s}U\frac{f_0 h'}{H}$$

Note that this is very similar to adding Rayleigh friction

$$\frac{\partial}{\partial t}q' + \frac{\partial}{\partial x}\left[Uq' + \beta\psi' + U\frac{f_0 h'}{H}\right] = -rq'$$

The waves must be damped away from the topography (or as $\mathbf{x}| \to \infty$ for finite time).

So what do damped waves look like away from the forcing ($U > 0$)? We have

$$\imath k\left[U - \frac{\beta}{k^2 + \ell^2}\right] = -r = \imath\omega(k)$$

For small r, $k = k_0 + \delta k$ and

$$\imath c_g \delta k = -r \quad \Rightarrow \quad \delta k = \imath r / c_g$$

So the waves will decay towards positive x: $\exp(\imath \delta k x) \simeq \exp(-xr/c_g)$. That's why we cannot have sinusoidal solutions upstream of the topography.