Flow over topography

Consider the following QG form for depth $H - h'$

$$
\frac{\partial}{\partial t} q + u \cdot \nabla q = 0
$$

$$
u = \hat{z} \times \nabla \psi
$$

$$
q = \nabla^2 \psi + \beta y + \frac{f_0}{H} h'
$$

For steady flow

$$
q = Q(\psi)
$$

and for an upstream flow $\psi \to -Uy$ and $h' = 0$, we have

$$
Q(\psi) = -\frac{\beta}{U} \psi
$$

which holds on all streamlines communicating with $\infty$. Substituting

$$
\nabla^2 \psi' + \frac{\beta}{U} \psi' = -\frac{f_0}{H} h'
$$

with $\psi = -U y + \psi'$. Note that solutions to this linear equation will still be solutions to the full problem.

We will deal with a simple case in a channel with width $W$

$$
h' = h(x) \sin \ell y \ , \ \ell = \frac{\pi}{W}
$$

and

$$
\psi' = \psi'(x) \sin \ell y
$$

Then

$$
\mathcal{L} \psi' = -\frac{f_0}{H} h'
$$

with the operator

$$
\mathcal{L} = \frac{\partial^2}{\partial x^2} - \ell^2 + \frac{\beta}{U}
$$
Greens’ functions

We can solve this kind of problem by defining
\[ \mathcal{L}G(x|x') = \delta(x - x') \]
giving
\[ \psi' = -\int dx' G(x|x') \frac{f_0}{H} h'(x') \]
For very narrow topography
\[ \psi' \simeq -\frac{f_0}{H} A G(x|x') \]
where \( A = \int dx' h'(x') \) being the area under the ridge. So \( G \) gives us a pretty good idea of the response.

Because the operator is second order, \( G \) must be continuous at \( x = x' \) but its first derivative will not be. With this operator
\[ G'(x' + \epsilon, x') - G'(x' - \epsilon, x') \to 1 \]
The Greens’ function can be written in terms of a solution \( \phi^+(x) \) which satisfies the boundary condition as \( x \to \infty \) and \( \phi^-(x) \) for the left boundary condition. Then
\[
G(x|x') = G_0 \begin{cases} 
\phi^-(x)\phi^+(x') & x < x' \\
\phi^-(x')\phi^+(x) & x > x'
\end{cases}
\]
and applying the matching condition on the derivative gives
\[ G_0 = 1/\left[ \phi^- \frac{\partial}{\partial x} \phi^+ - \phi^+ \frac{\partial}{\partial x} \phi^- \right] \]
if \( U < 0 \) we have
\[ \phi^\pm = \exp(\mp kx) \]
with
\[ k = \sqrt{\beta U} \]
and
\[ G = -\frac{1}{2k} \exp(-k|x - x'|) \]
But for \( U > 0 \) there can be sinusoidal waves with
\[ k = \sqrt{\beta U - \ell^2} \]
and
\[ \phi^- = 0 , \quad \phi^+ = \sin(kx) \]
giving
\[
G(x|x') = \begin{cases} 
0 & x < x' \\
\frac{1}{k} \sin k(x - x') & x > x'
\end{cases}
\]
and we have a downstream lee wave with
\[ \psi = -\frac{f_0 A}{Hk} \sin kx \]
Lee wave

Why the asymmetry? Why not sines or cosines for $x < x'$ as well? Here we cannot apply the boundary condition $\psi' \to 0$; we need a radiation condition. No energy should be coming in from $-\infty$ and it should be going out at $+\infty$. 

**Group velocity**

\[ \omega = Uk - \frac{\beta k}{k^2 + \ell^2} \]

gives

\[ c_g = U + \beta \frac{k^2 - \ell^2}{(k^2 + \ell^2)^2} \]

(westward for long waves, eastward for short waves when $U = 0$). For the stationary wave, though

\[ c_g = U + U \frac{k^2 - \ell^2}{k^2 + \ell^2} = U \frac{2k^2}{k^2 + \ell^2} > 0 \]

So waves on the left side would not satisfy the radiation condition.

**Initial value problem**

For the linearized initial value problem

\[ \frac{\partial}{\partial t} q' + \frac{\partial}{\partial x} \left[ Uq' + \beta \psi' + U \frac{f_0 h'}{H} \right] = 0 \]

with

\[ \left[ \frac{\partial^2}{\partial x^2} - \ell^2 \right] \psi' = q' \]

If we write the Laplace transform solution

\[ \hat{q}(x) = \int_0^\infty e^{-st} q(x, t) \]

then

\[ s\hat{q} + \frac{\partial}{\partial x} \left[ U\hat{q} + \beta \hat{\psi} \right] = -\frac{1}{s} U \frac{f_0 h'}{H} \]

Note that this is very similar to adding Rayleigh friction

\[ \frac{\partial}{\partial t} q' + \frac{\partial}{\partial x} \left[ Uq' + \beta \psi' + U \frac{f_0 h'}{H} \right] = -rq' \]

The waves must be damped away from the topography (or as $x \to \infty$ for finite time). 

So what do damped waves look like away from the forcing ($U > 0$)? We have

\[ ik \left[ U - \frac{\beta}{k^2 + \ell^2} \right] = -r = \omega(k) \]

For small $r$, $k = k_0 + \delta k$ and

\[ uc_g \delta k = -r \quad \Rightarrow \quad \delta k = \frac{r}{c_g} \]

So the waves will decay towards positive $x$: $\exp(i\delta k x) \simeq \exp(-rx/c_g)$. That’s why we cannot have sinusoidal solutions upstream of the topography.